Angles and circles

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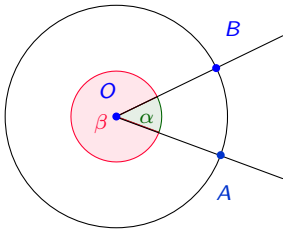
Central angle

Definition

An angle with a vertex in the center of the circle we called the central angle.



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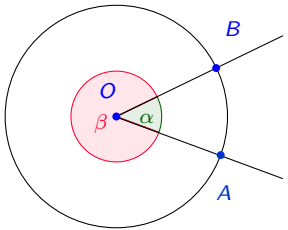

Picture 1.

The arms (sides) of this angle contain two different radii of the circle, so $\alpha, \beta \in (0°, 360°)$.

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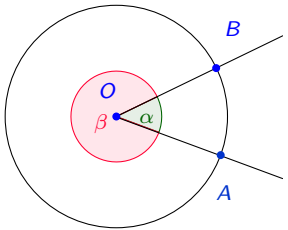
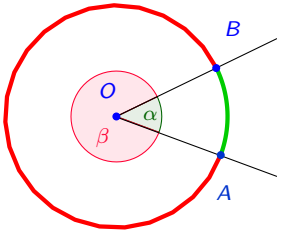


Picture 1.

The arms (sides) of this angle contain two different radii of the circle, so $\alpha, \beta \in (0°, 360°)$.
α is a convex angle and β is a concave angle.

Intercepted arc

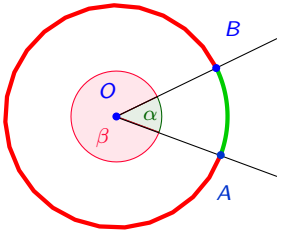


Picture 2.

Definition

The arcs „cut off" or „lying between" the sides of this angles we called intercepted arcs.

Intercepted arc



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Definition

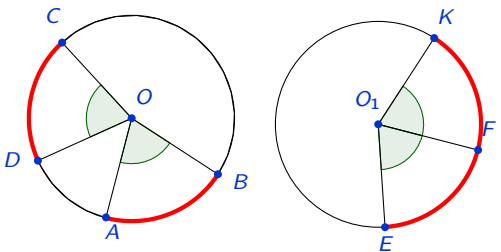
The arcs „cut off" or „lying between" the sides of this angles we called intercepted arcs.

α's intercepted arc is the green arc from A to B
β's intercepted arc is the red arc from A to B.

Central angles theorem

Theorem

In a circle (or congruent circles) the central angles are congruent if and only if their intercepted arcs are congruent.

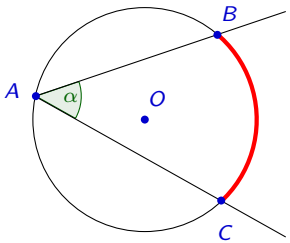


Picture 3.

Inscribed angle

Definition

An inscribed angle is an angle with its vertex on the circle. The sides of this angle contain two different chords of the circle.



Picture 4.

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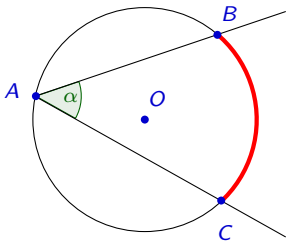

Picture 4.

$\alpha \in (0°, 180°)$

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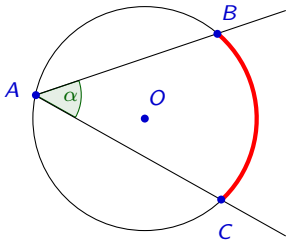

Picture 4.

$\alpha \in (0°, 180°)$

We say that α is subtended by the red arc BC or the red arc is its intercepted arc.

Central and inscribed angles theorem

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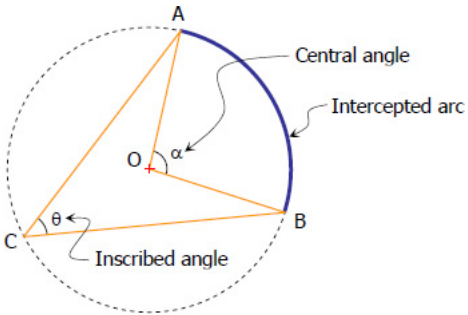
Theorem

The central angle of a circle is exactly twice the measure of an inscribed angle subtended by the same arc.



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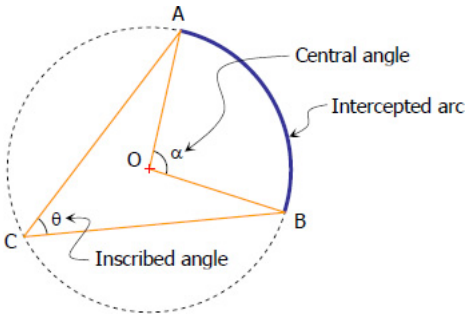

This single property has a lot of consequences.

Proof – The first case

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The center of the circle lays on the side of inscribed angle

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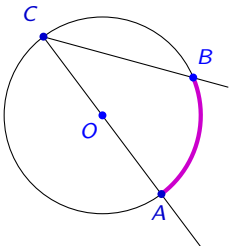

Picture 5.

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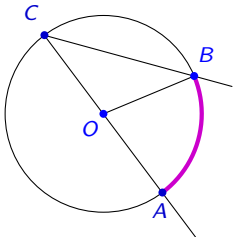

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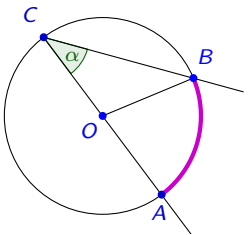

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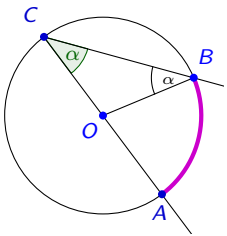

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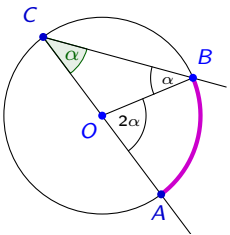

Picture 5.

Central angle $\sphericalangle BOA$ is an exterior angle of a triangle BOC.

Proof – The second case

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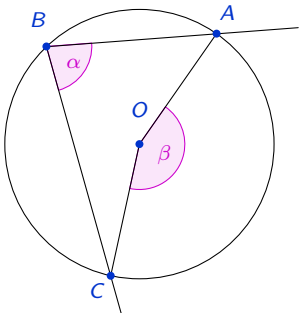
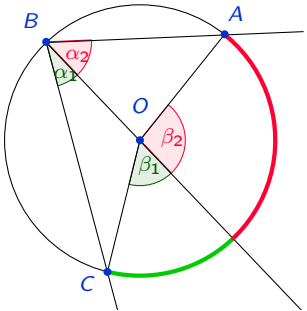


Picture 6.

Proof – The second case

The center of the circle is in the interior of inscribed angle



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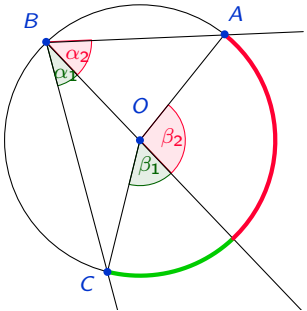

Picture 6.

$$\alpha = \alpha_1 + \alpha_2 \qquad \text{and} \qquad \beta = \beta_1 + \beta_2$$

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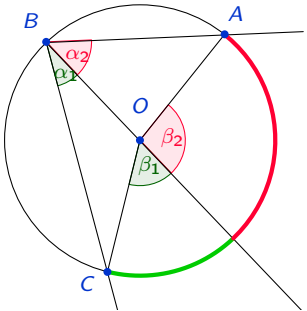

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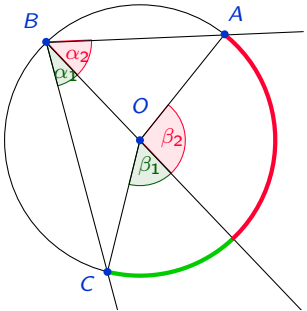

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Proof – The third case

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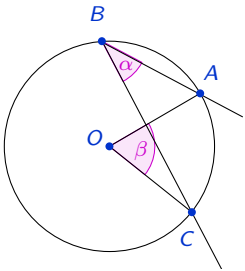

Picture 7.

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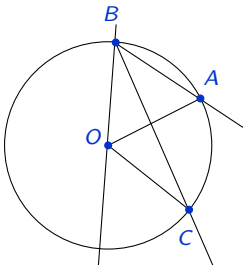

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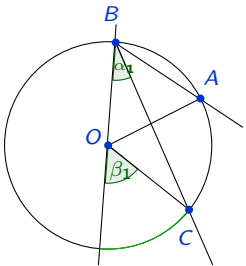

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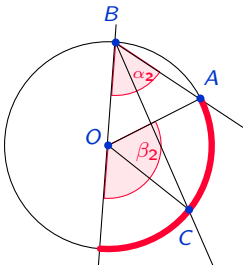

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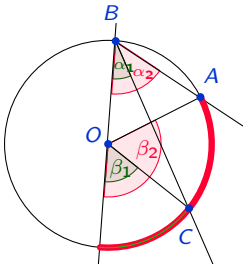

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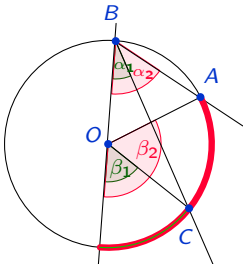

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$$\alpha = \alpha_2 - \alpha_1 \qquad \text{and} \qquad \beta = \beta_2 - \beta_1$$

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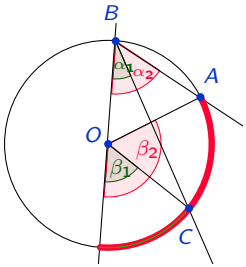

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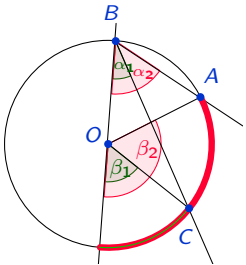

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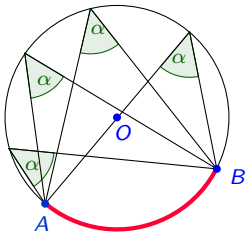
Corollary (1)

Inscribed angles subtended by the same arc are congruent.



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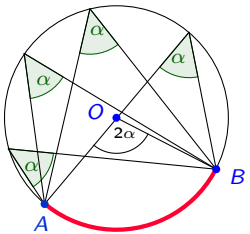

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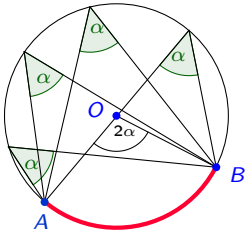


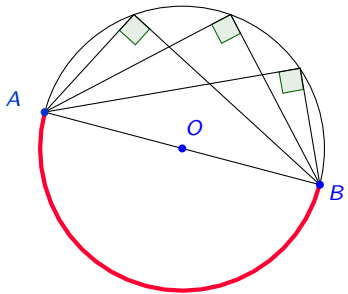
Picture 8.

Corollary (2)

Inscribed angles subtended by arcs of the same length are congruent.

Corollary (3)

An inscribed angle subtended by a diameter (by semi-circle) is always 90°, i.e., a right angle.



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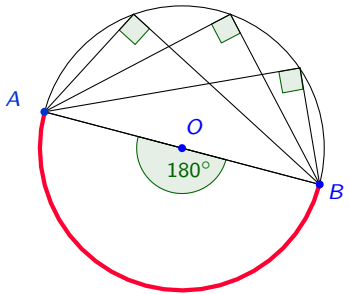

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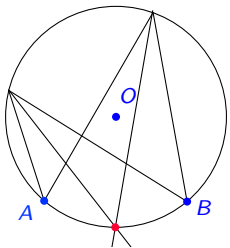
Corollary (4)

Bisectors of inscribed angles subtended by the same arc intersect at the midpoint of this arc.

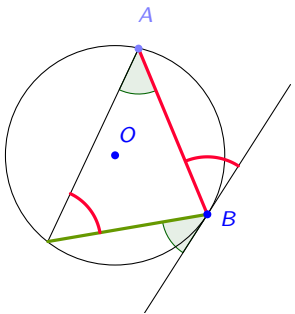


Picture 10.

Alternate Segment Theorem

Theorem

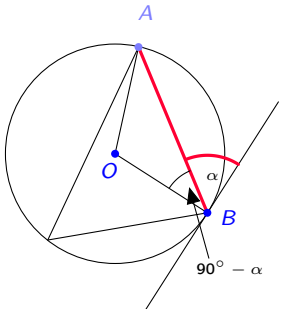
The angle between a tangent and a chord drawn from the point of contact is equal to any angle subtended by the chord in the alternate segment.



Picture 11.

Proof

We draw the radii OA and OB.
A tangent to a circle is perpendicular to the radius drawn to the point of tangency, so $|\sphericalangle OBA| = 90^\circ - \alpha$.



Picture 12.

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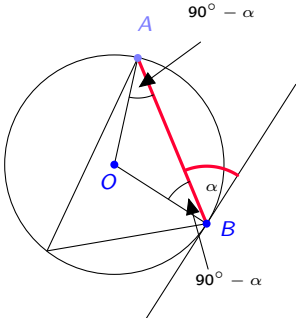
We draw the radii OA and OB. We know that $|\sphericalangle OBA| = 90^\circ - \alpha$.



Picture 12.

Triangle AOB is isosceles, so $|\sphericalangle OAB| = 90^\circ - \alpha$.

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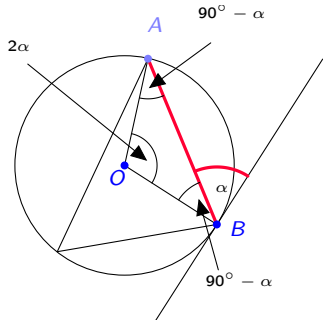

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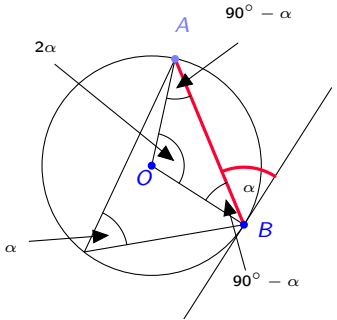
Proof

We draw the radii OA and OB. We know that $|\sphericalangle OBA| = 90^\circ - \alpha$.



Picture 12.

Triangle AOB is isosceles, so $|\sphericalangle OAB| = 90^\circ - \alpha$.

Problem 1

Look at the picture

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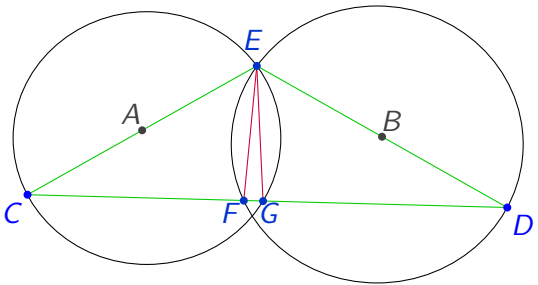
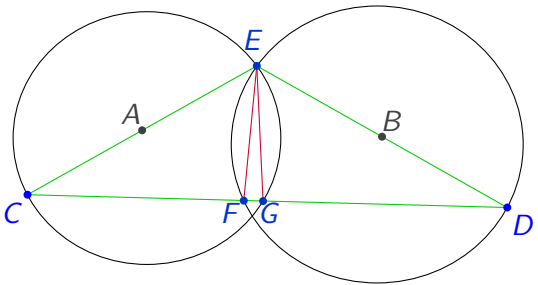


Problem 1

Look at the picture

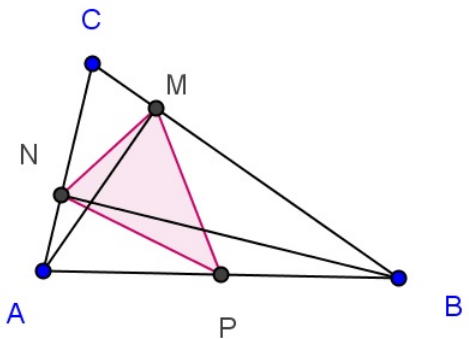


Triangle *EFG* has two right angles!

Problem 2

Problem

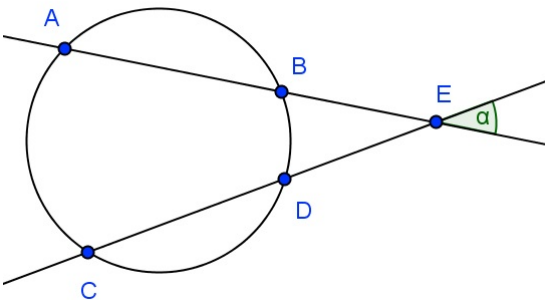
Let us consider the acute-angled triangle ABC where the angle C is 60°*. Segments AM and BN are altitudes of this triangle and P is the center of the segment AB. Prove that the triangle MNP is equilateral.*



Problem 3

Problem

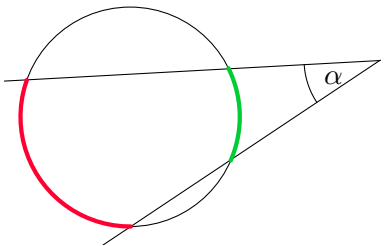
Compute the angle BCD knowing that $\alpha = 40°$, *and* $|AB| = |AC| = |CD|$.



Problem 4

Problem

The angle α is formed by two secants intersecting outside of circle. Count the measure of this angle if you know that the measure of central angles subtended by red and green arc are equal α and β respectively.



Problem 5

Problem

Quadrilateral ABCD is inscribed in a circle. There are no pairs of parallel sides. The extensions of sides AB and CD intersect at point K, and the sides AD and BC at L. Prove that the bisectors of the angles with vertex K and L are perpendicular.