Motivation

How to determine a right angle having only a cord and three nails?

Pythagorean Theorem

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Summer math camp in Zakopane, 2024

Topics

Lecture 1 Various proofs of the Pythagorean Theorem.

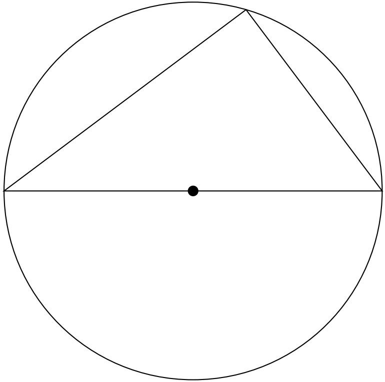
Lecture 2 Some generalizations and consequences of the Pythagorean Theorem.

Lecture 3 Examples of interesting problems solved using Pythagorean Theorem.

Main tools

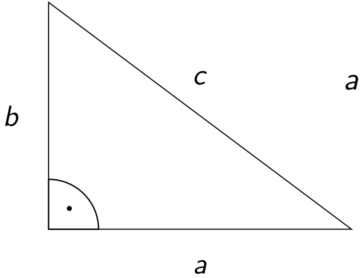
- Pythagorean Theorem and The converse theorem
- Similarity of figures
- The inscribed angle is a right angle iff it is inscribed in a semicircle.

  The triangle is a right triangle iff the center of the described circle is in the midpoint of one of the sides of this triangle.



The Pythagorean Theorem

In a right triangle: the squares of the lengths of the legs is equal to the square of the length of the hypothenuse.



$$a^2 + b^2 = c^2$$

The converse theorem

A triangle with the sides of the lengths a, b and c, where $a < c$, $b < c$ is given.
If $a^2 + b^2 = c^2$, then this triangle is a right triangle.

For example: the triangle with sides 3, 4, 5 is a right triangle.

Such triples of numbers are called Pythagorean triples.

Another examples of Pythagorean triples: 5, 12, 13 ....

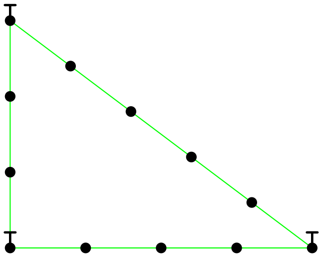
Pythagorean triples

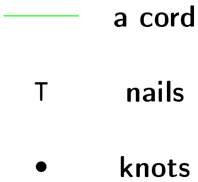
3, 4, 5: ancient Egypt, ancient China

5, 12, 13: ancient Hindu world

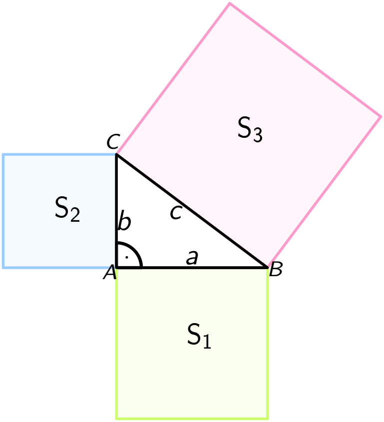
How to determine a right angle having only a cord and three nails?

The cord should be tied and should have 12 knots.





Geometric interpretation

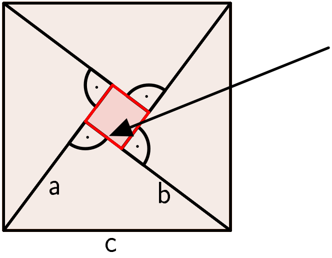


$$S_1 = a^2$$
$$S_2 = b^2$$
$$S_3 = c^2$$

$$a^2 + b^2 = c^2$$

$$S_1 + S_2 = S_3$$

Proof 1



the side of the red square $= b - a$

the area of the red square $= (b-a)^2$

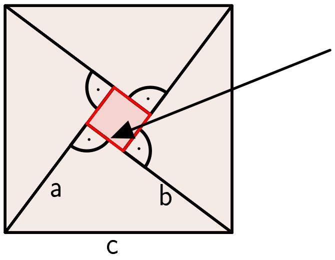
the area of the big square $= c^2$

the area of each of four right triangle

$= \frac{1}{2}ab$

$$\underline{c^2} = 4 \cdot \frac{1}{2}ab + (b-a)^2 = 2ab + b^2 - 2ab + a^2 = \underline{b^2 + a^2}$$

Proof 1



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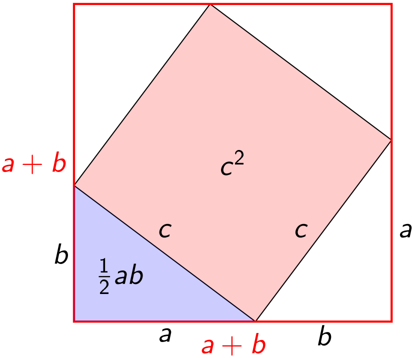
$$\underline{c^2} = 4 \cdot \frac{1}{2}ab + (b-a)^2 = 2ab + b^2 - 2ab + a^2 = \underline{b^2 + a^2}$$

$$c^2 = a^2 + b^2$$

Proof 2

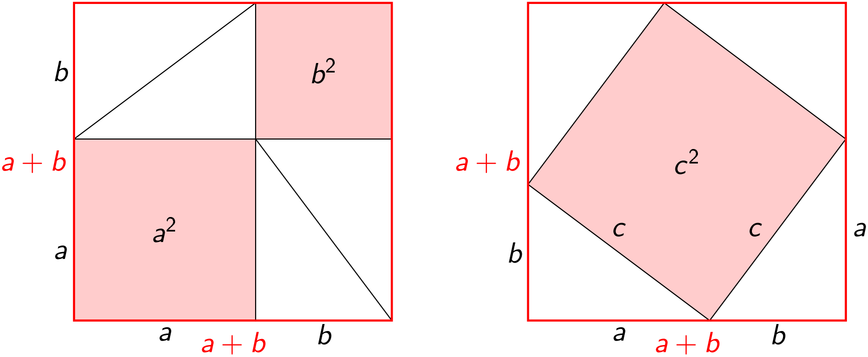


$$\text{the area of the big square } = (a+b)^2 = \underline{a^2 + 2ab + b^2}$$

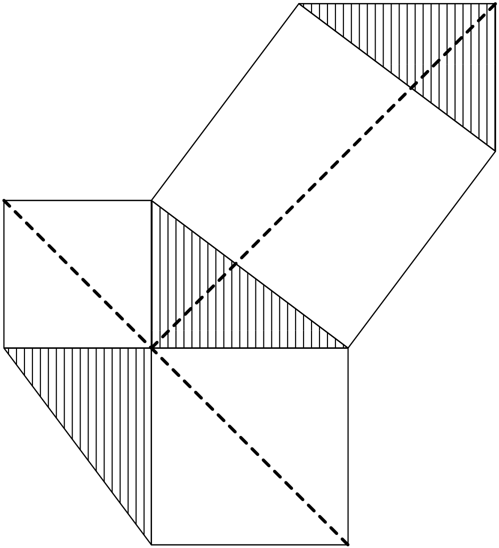
$$4 \cdot \frac{1}{2}ab + c^2 = \underline{2ab + c^2}$$

$$a^2 + b^2 = c^2$$

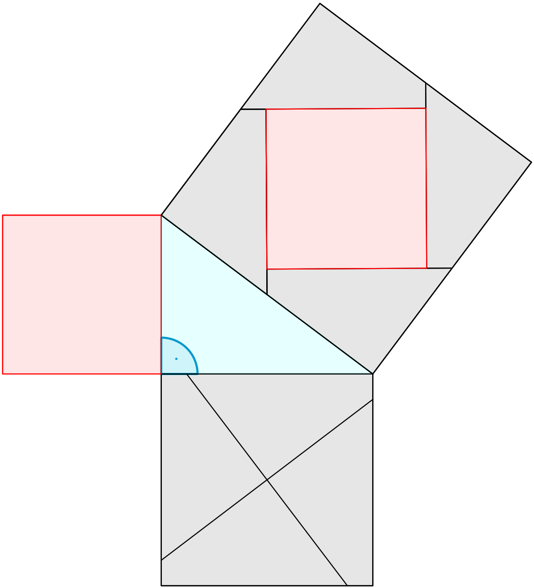
Proof 3



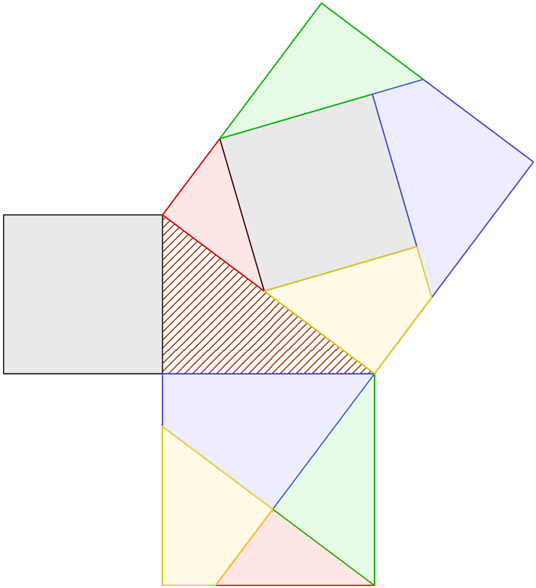
Proof 4, Leonardo da Vinci (1452 - 1519)



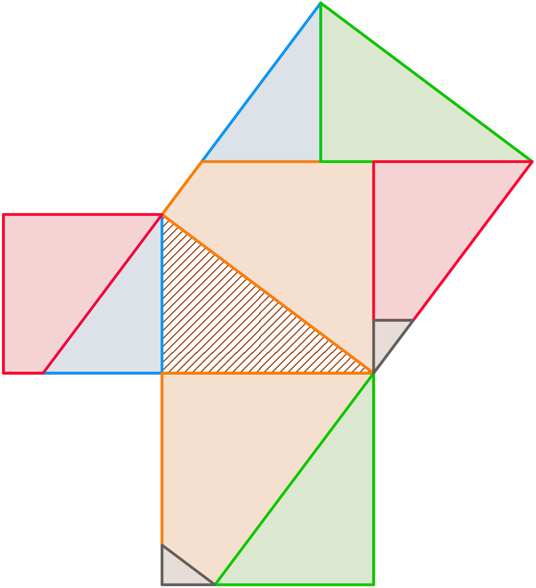
Proof 5, Henri Perigal (1873)



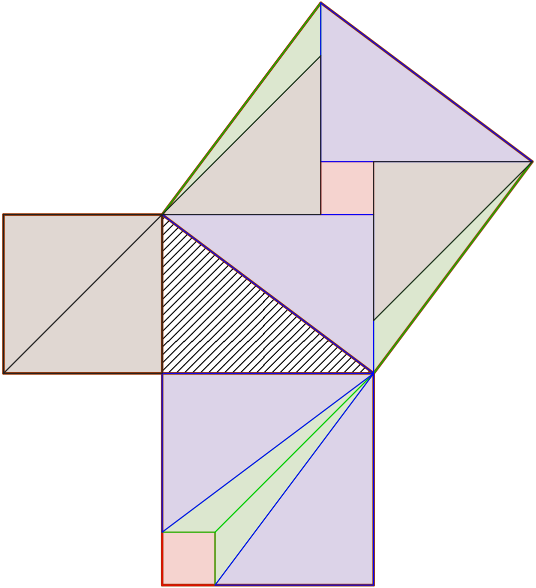
Proof 6 (yellow puzzle)



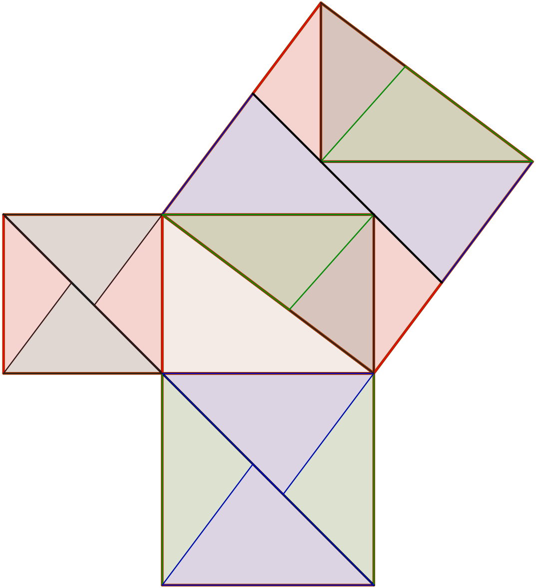
Proof 7 (green puzzle)



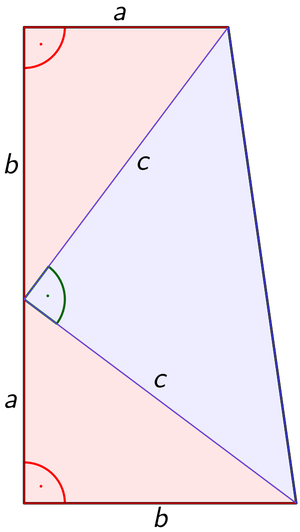
Proof 8, Liu Hui (3rd century A.D.)



Proof 9 (blue puzzle), J. E. Böttcher



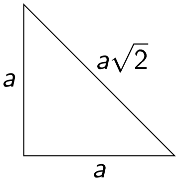
Proof 10 (red puzzle), James A. Garfield (1876) - 20th President of the USA



Generalizations and consequences

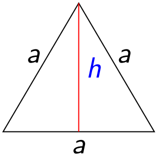
Special triangles

Triangle $90^\circ, 45^\circ, 45^\circ$



$$area = \frac{1}{2}a^2$$

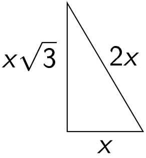
Triangle $60^\circ, 60^\circ, 60^\circ$



$$h = \frac{a\sqrt{3}}{2}$$

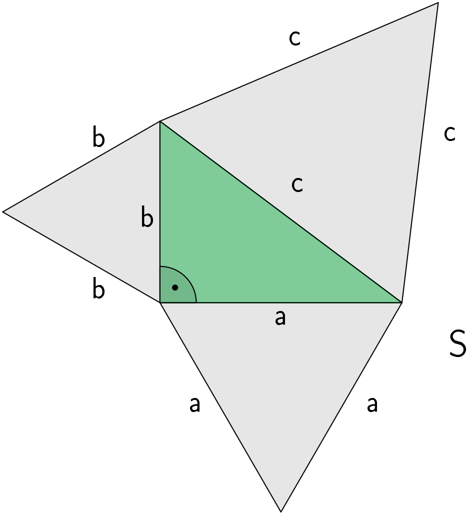
$$area = \frac{a^2\sqrt{3}}{4}$$

Triangle $90^\circ, 60^\circ, 30^\circ$



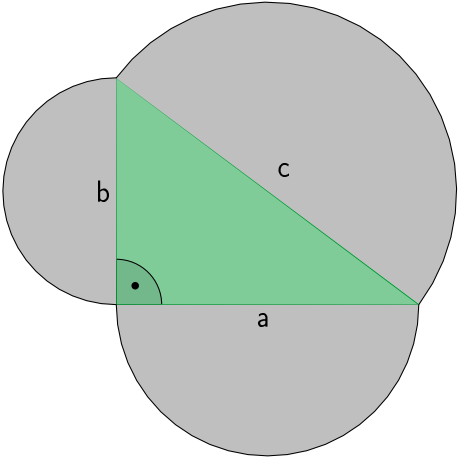
$$area = \frac{x^2\sqrt{3}}{2}$$

Generalizations



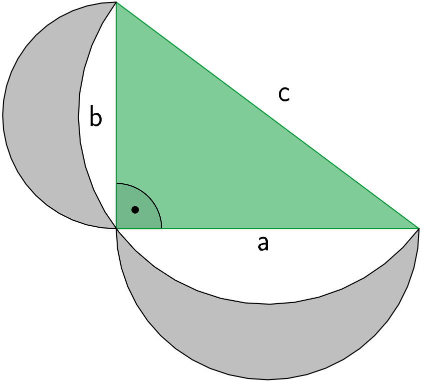
$$S_1 + S_2 = S_3 \ ?$$

Generalizations



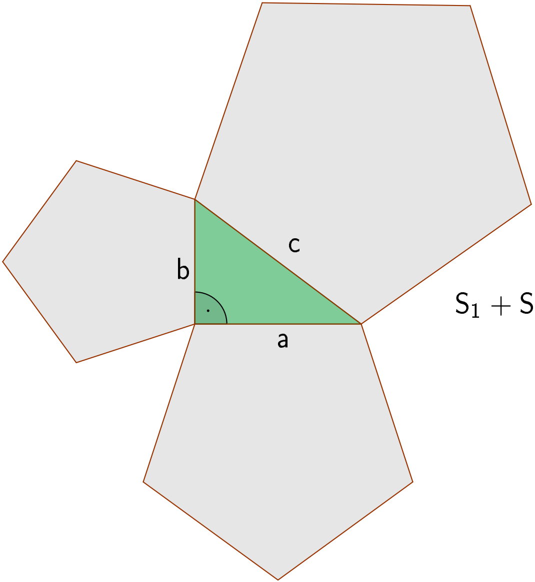
$$S_1 + S_2 = S_3 \ ?$$

The lune of Hippocrates



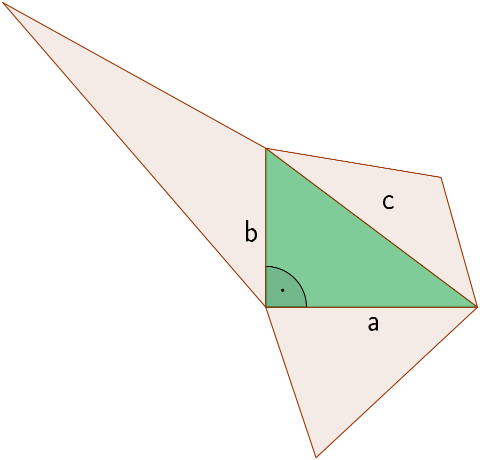
$S_{shaded\ figure} = ?$

Generalizations



$S_1 + S_2 = S_3$?

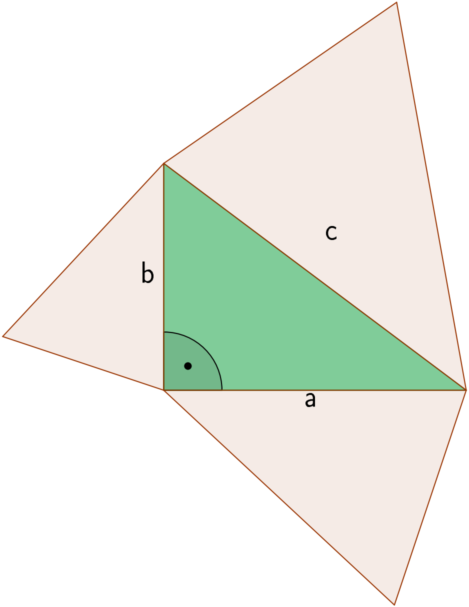
Generalizations



$$S_1 + S_2 = S_3 \ ?$$

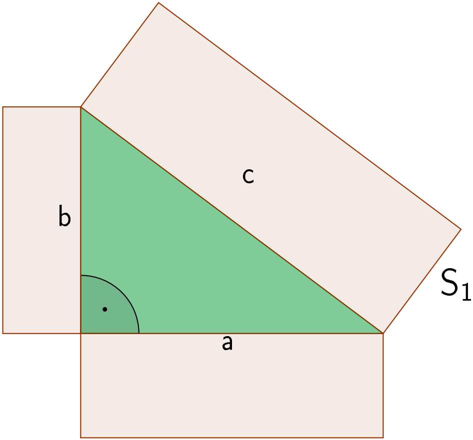
If figures are similar,
then the equality holds !

Generalizations



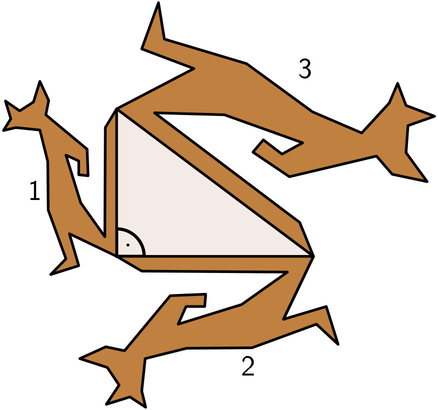
$$S_1 + S_2 = S_3$$

Generalizations



$$S_1 + S_2 = S_3$$

Generalizations



S_1 – the area of kangaroo 1

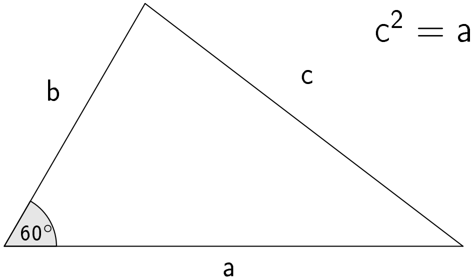
S_2 – the area of kangaroo 2

S_3 – the area of kangaroo 3

If the figures : 1, 2, and 3 are similar then

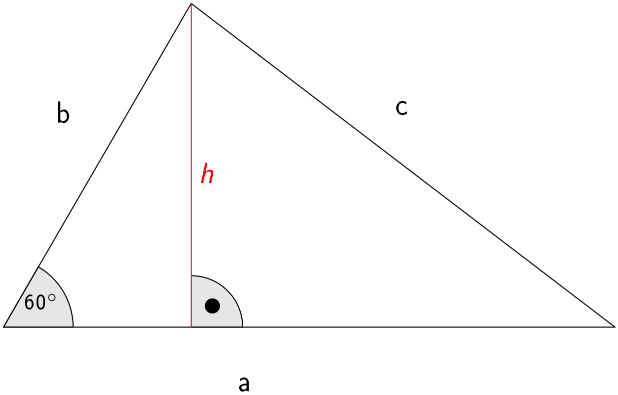
$$S_1 + S_2 = S_3$$

Consequences

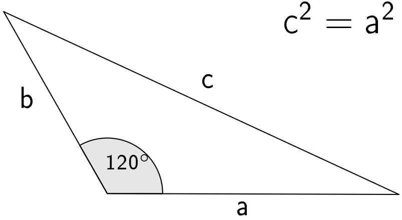


$$c^2 = a^2 + b^2 - ab$$

Consequences

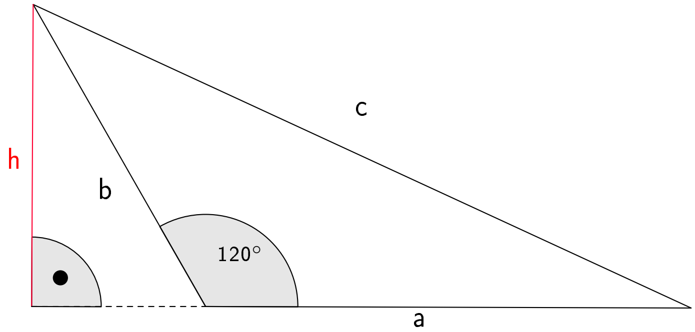


Consequences



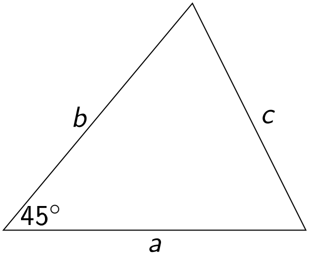
$$c^2 = a^2 + b^2 + ab$$

Consequences



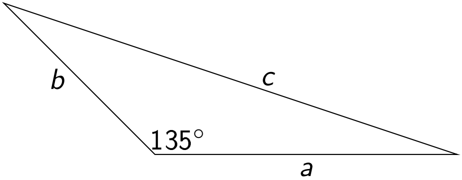
Exercises

Ex. 1



$$c^2 = ?$$

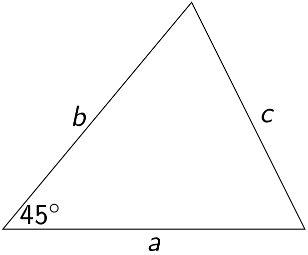
Ex. 2



$$c^2 = ?$$

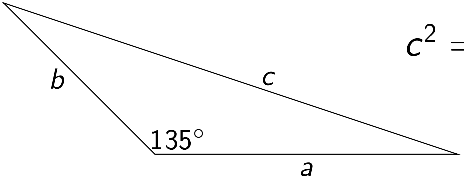
Exercises

Ex. 1



$$c^2 = a^2 + b^2 - ab\sqrt{2}$$

Ex. 2



$$c^2 = a^2 + b^2 + ab\sqrt{2}$$

Observation

For angle 60°: $c^2 = a^2 + b^2 - ab$
For angle 45°: $c^2 = a^2 + b^2 - ab\sqrt{2}$

Observation

For angle $60°$: $c^2 = a^2 + b^2 - ab$
For angle $45°$: $c^2 = a^2 + b^2 - ab\sqrt{2}$

$$c^2 < a^2 + b^2$$

Observation

For angle 60°: $c^2 = a^2 + b^2 - ab$
For angle 45°: $c^2 = a^2 + b^2 - ab\sqrt{2}$

$$c^2 < a^2 + b^2$$

For angle 120°: $c^2 = a^2 + b^2 + ab$
For angle 135°: $c^2 = a^2 + b^2 + ab\sqrt{2}$

Observation

For angle 60°: $c^2 = a^2 + b^2 - ab$
For angle 45°: $c^2 = a^2 + b^2 - ab\sqrt{2}$

$$c^2 < a^2 + b^2$$

For angle 120°: $c^2 = a^2 + b^2 + ab$
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$$c^2 > a^2 + b^2$$

Observation

For angle 60°: $c^2 = a^2 + b^2 - ab$
For angle 45°: $c^2 = a^2 + b^2 - ab\sqrt{2}$

$$c^2 < a^2 + b^2$$

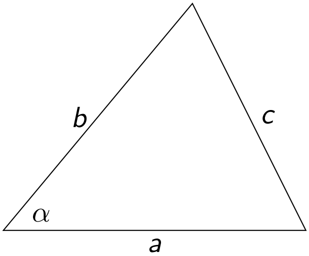
For angle 120°: $c^2 = a^2 + b^2 + ab$
For angle 135°: $c^2 = a^2 + b^2 + ab\sqrt{2}$

$$c^2 > a^2 + b^2$$

It is not a coincidence!

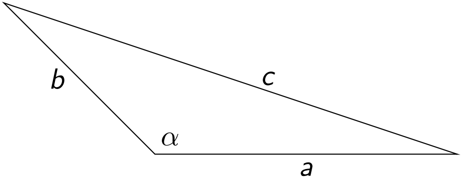
Facts

- $\alpha < 90°$



$$c^2 < a^2 + b^2$$

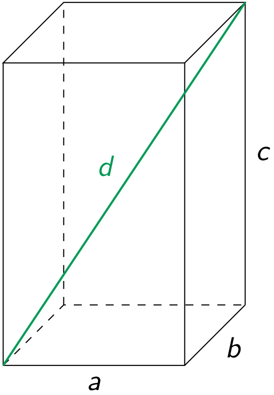
- $\alpha > 90°$



$$c^2 > a^2 + b^2$$

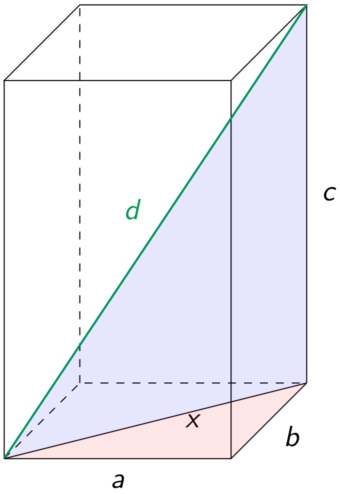
3D versions

The square of the lenght of the diagonal of the cuboid is equal to the sum of the squares of the lenghts of its edges.



$$d^2 = a^2 + b^2 + c^2$$

3D versions



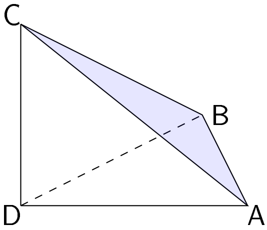
$$\underbrace{x^2 = a^2 + b^2}$$

$$d^2 = x^2 + c^2$$

$$d^2 = a^2 + b^2 + c^2$$

3D versions

In the rectangular tetrahedron with the three right angles in vertex D, the sum of squares of the areas of the three faces with vertex D is equal to square of the area of the fourth face.



If

$$|\sphericalangle ADC| = |\sphericalangle BDC| = |\sphericalangle ADB| = 90^\circ,$$

then

$$S_{ABC}^2 = S_{ADC}^2 + S_{BDC}^2 + S_{ADB}^2.$$

Problems

60°	$c^2 = a^2 + b^2 - ab$	120°	$c^2 = a^2 + b^2 + ab$
45°	$c^2 = a^2 + b^2 - ab\sqrt{2}$	135°	$c^2 = a^2 + b^2 + ab\sqrt{2}$

Problem 1. Prove that for each positive a, b, c

$$\sqrt{a^2+b^2} + \sqrt{b^2+c^2} + \sqrt{a^2+c^2} \geq \sqrt{2}(a+b+c)$$

60°	$c^2 = a^2 + b^2 - ab$
45°	$c^2 = a^2 + b^2 - ab\sqrt{2}$
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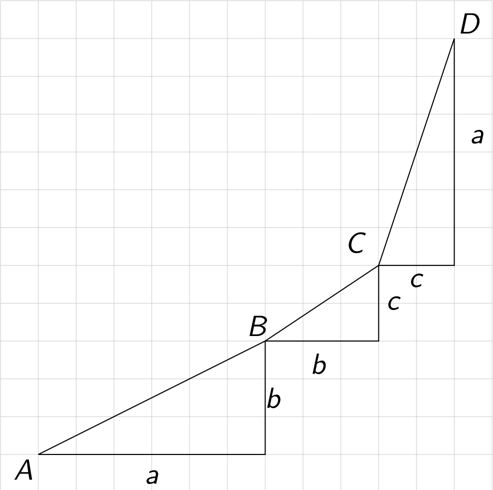
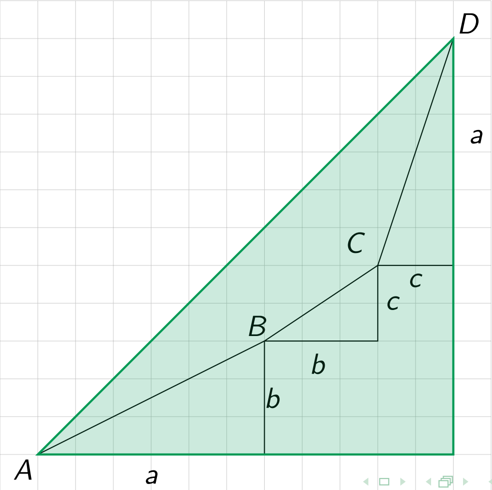


$\boxed{60^\circ}$ $\boxed{c^2 = a^2 + b^2 - ab}$ $\boxed{120^\circ}$ $\boxed{c^2 = a^2 + b^2 + ab}$

$\boxed{45^\circ}$ $\boxed{c^2 = a^2 + b^2 - ab\sqrt{2}}$ $\boxed{135^\circ}$ $\boxed{c^2 = a^2 + b^2 + ab\sqrt{2}}$

Problem 1. Prove that for each positive a, b, c

$$\sqrt{a^2+b^2} + \sqrt{b^2+c^2} + \sqrt{a^2+c^2} \geq \sqrt{2}(a+b+c)$$

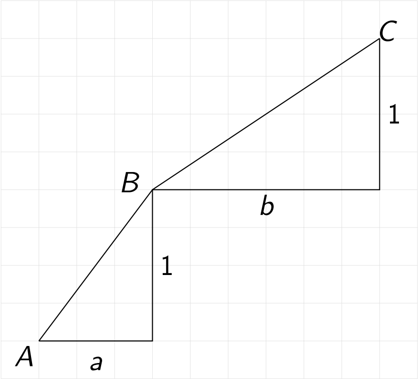


Problem 2. Prove that for each positive a, b:
$\sqrt{a^2+1} + \sqrt{b^2+1} \geq 2\sqrt{a+b}$

$\boxed{60^\circ}$ $\boxed{c^2 = a^2 + b^2 - ab}$ $\boxed{120^\circ}$ $\boxed{c^2 = a^2 + b^2 + ab}$

$\boxed{45^\circ}$ $\boxed{c^2 = a^2 + b^2 - ab\sqrt{2}}$ $\boxed{135^\circ}$ $\boxed{c^2 = a^2 + b^2 + ab\sqrt{2}}$

Problem 2. Prove that for each positive a, b:
$\sqrt{a^2+1} + \sqrt{b^2+1} \geq 2\sqrt{a+b}$



60°	$c^2 = a^2 + b^2 - ab$	120°	$c^2 = a^2 + b^2 + ab$
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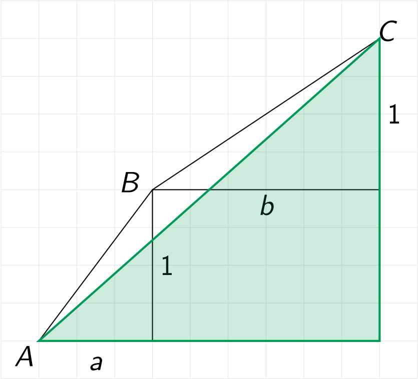
Problem 2. Prove that for each positive a, b:
$\sqrt{a^2+1} + \sqrt{b^2+1} \geq 2\sqrt{a+b}$



Exercise. Prove that for each positive a, b, c:

$$\sqrt{a^2+1} + \sqrt{b^2+1} + \sqrt{c^2+1} \geq \sqrt{6(a+b+c)}$$

$\boxed{60^\circ}$ $\boxed{c^2 = a^2 + b^2 - ab}$	$\boxed{120^\circ}$ $\boxed{c^2 = a^2 + b^2 + ab}$
$\boxed{45^\circ}$ $\boxed{c^2 = a^2 + b^2 - ab\sqrt{2}}$	$\boxed{135^\circ}$ $\boxed{c^2 = a^2 + b^2 + ab\sqrt{2}}$

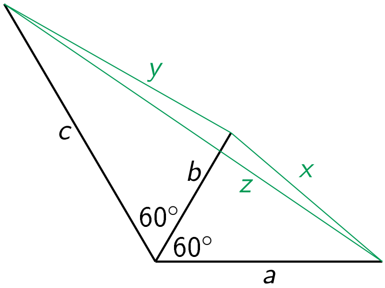
Problem 3. Prove that for each positive a, b, c:
$$\sqrt{a^2 + b^2 - ab} + \sqrt{b^2 + c^2 - bc} \geq \sqrt{a^2 + c^2 + ac}$$

60°	$c^2 = a^2 + b^2 - ab$
45°	$c^2 = a^2 + b^2 - ab\sqrt{2}$

120°	$c^2 = a^2 + b^2 + ab$
135°	$c^2 = a^2 + b^2 + ab\sqrt{2}$

Problem 3. Prove that for each positive a, b, c:

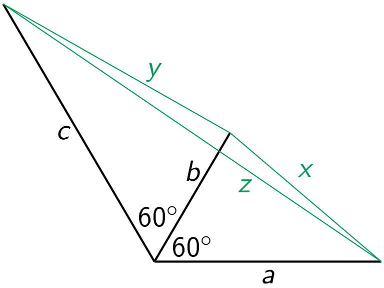
$$\sqrt{a^2 + b^2 - ab} + \sqrt{b^2 + c^2 - bc} \geq \sqrt{a^2 + c^2 + ac}$$



60°	$c^2 = a^2 + b^2 - ab$	120°	$c^2 = a^2 + b^2 + ab$
45°	$c^2 = a^2 + b^2 - ab\sqrt{2}$	135°	$c^2 = a^2 + b^2 + ab\sqrt{2}$

Problem 3. Prove that for each positive a, b, c:

$$\sqrt{a^2 + b^2 - ab} + \sqrt{b^2 + c^2 - bc} \geq \sqrt{a^2 + c^2 + ac}$$



$$x + y \geq z$$

$\boxed{60^\circ}$ $\boxed{c^2 = a^2 + b^2 - ab}$ $\boxed{120^\circ}$ $\boxed{c^2 = a^2 + b^2 + ab}$

$\boxed{45^\circ}$ $\boxed{c^2 = a^2 + b^2 - ab\sqrt{2}}$ $\boxed{135^\circ}$ $\boxed{c^2 = a^2 + b^2 + ab\sqrt{2}}$

Exercise. Prove that for each positive a, b:

$$\sqrt{a^2 + b^2 + ab\sqrt{2}} - \sqrt{a^2 + b^2 - ab\sqrt{2}} < a\sqrt{2}$$

Problem 4. Prove that for each positive a, b, c such that $a^2 + b^2 - ab = c^2$ we have $(a - c)(b - c) \leq 0$.

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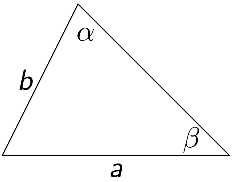
Fact: In an arbitrary triangle the length of the side opposite the smaller angle is smaller than the length of the side opposite the bigger angle.



$b < a$ iff $\beta < \alpha$

Exercise. Prove that for each positive a, b, c such that $a^2 + b^2 + ab = c^2$ we have $(a - c)(b - c) > 0$.

Problem 5. Prove that for each positive a, b, c such that $a > c$ and $b > c$ we have $\sqrt{c(a-c)} + \sqrt{c(b-c)} \leq \sqrt{ab}$.

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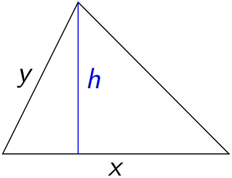
Fact: In an arbitrary triangle which two of sides has the length x and y, we have $S_\triangle \leq \frac{1}{2}xy$.



$$h \leq y$$

$$S_\triangle = \frac{1}{2}xh \leq \frac{1}{2}xy$$

Problem 5. Prove that for each positive a, b, c such that $a > c$ and $b > c$ we have $\sqrt{c(a-c)} + \sqrt{c(b-c)} \leq \sqrt{ab}$.

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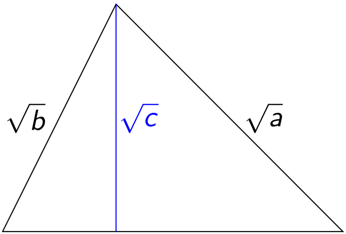

Exercise. Prove that for each positive $a, b > 1$ we have

$$\sqrt{a^2 - 1} + \sqrt{b^2 - 1} \leq ab.$$

Problem 6. Determine the value of the expression $2ab + c(a+b)\sqrt{2}$, where $a, b, c > 0$ satisfy all of the following conditions:

$$\begin{cases} a^2 + b^2 = 9 \\ a^2 + c^2 + ac\sqrt{2} = 16 \\ b^2 + c^2 + bc\sqrt{2} = 25 \end{cases}$$