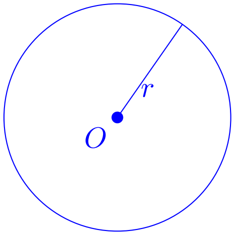
Convexity

Witold Kraśkiewicz

Zakopane 2024

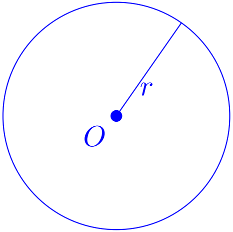
Terminology

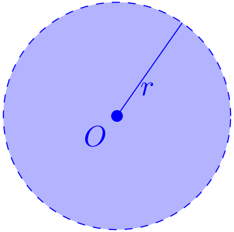


circle

$$\{ P : |OP| = r \}$$

Terminology





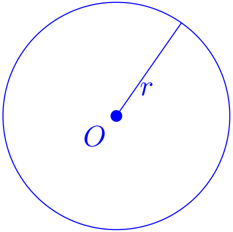
circle

$\{ P : |OP| = r \}$

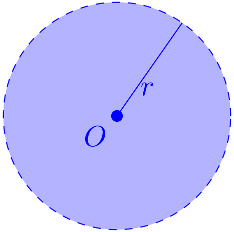
open disc

$\{ P : |OP| < r \}$

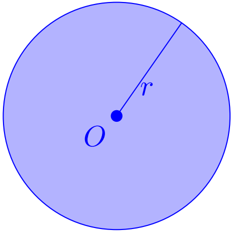
Terminology



circle
$\{\, P : |OP| = r \,\}$



open disc
$\{\, P : |OP| < r \,\}$



(closed) disc
$D(P, r) =$
$\{\, P : |OP| \leq r \,\}$

Terminology, cnt.

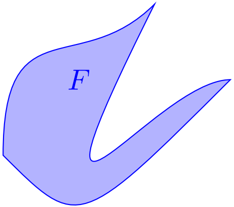
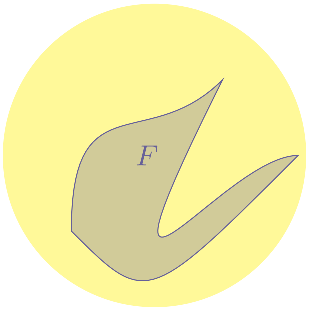


Figure F is bounded iff there exists a point P and a number $r > 0$ such that

$$F \subset D(P, r).$$

Terminology, cnt.



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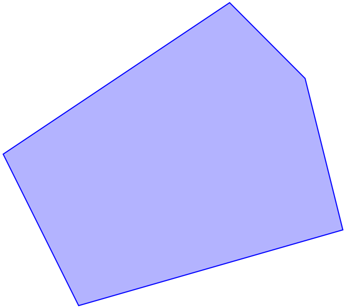
Terminology, cnt.

Let F be a figure.



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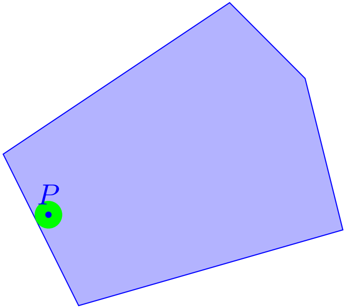

Point P is **inner point** of F iff there exists $r > 0$ such that $D(P, r) \subset F$.

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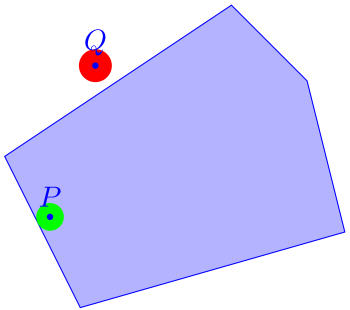

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Point Q is **outer point** of F iff there exists $r > 0$ such that $D(Q, r) \cap F = \emptyset$.

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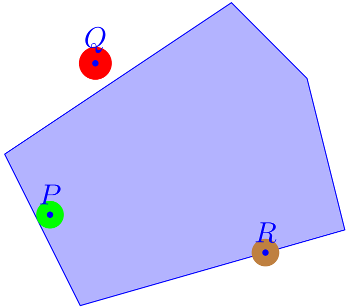

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Point R is **boundary point** of F iff it is neither inner nor outer point of F.

Terminology, cnt.

Boundary of a figure is the set off all its boundary points.

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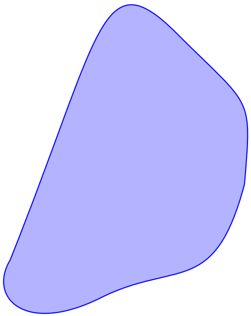
A geometric figure is said to be **compact** if it is bounded and closed.

Convexity

Figure F is convex iff
given any two points $A, B \in F$, line segment AB is contained in F.



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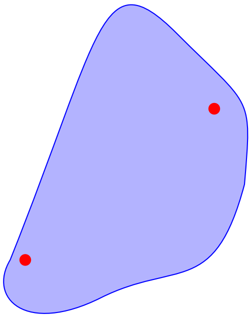

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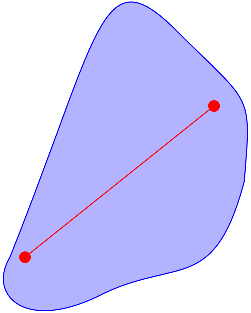
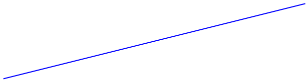


Convex sets on straight line



line

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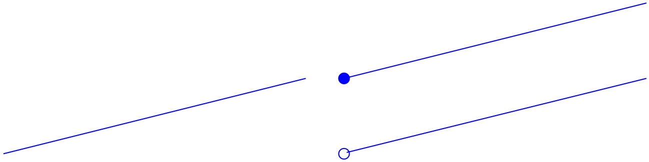
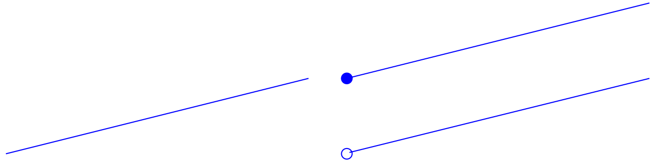


line half-lines

Convex sets on straight line



line

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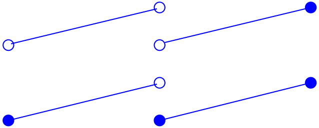
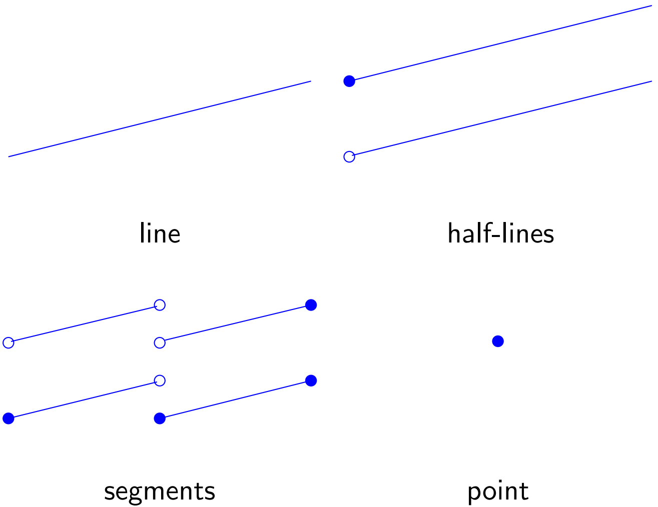


segments

Convex sets on straight line



Convex sets, examples

- half-plane

Convex sets, examples

- half-plane
- convex angle

Convex sets, examples

- half-plane
- convex angle
- convex polygon

Convex sets, examples

- half-plane
- convex angle
- convex polygon
- disc

Properties of convex sets

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For any X, $\operatorname{conv}(X)$ is a convex set.
It is the smallest convex set containing X.
$\operatorname{conv}(X)$ is called convex hull of X.

Convex hull, examples

Helly's theorem

Let $X_1, X_2, \ldots, X_n$, $n \geq 3$, be a finite collection of convex figures. If the intersection of every 3 of these figures is nonempty, then the whole collection has a nonempty intersection.

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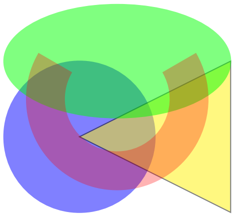
Convexity assumption is essential!



Helly's theorem, special case

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Proof:

$Y_{123} = X_1 \cap X_2 \cap X_3 \quad Y_{124} = X_1 \cap X_2 \cap X_4 \quad Y_{134} = X_1 \cap X_3 \cap X_4 \quad Y_{234} = X_2 \cap X_3 \cap X_4$

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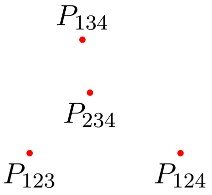
case (1): one of the P_I's lies in the convex hulls of the others



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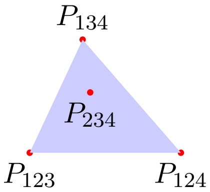

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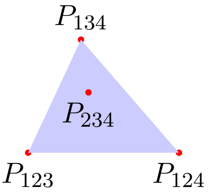

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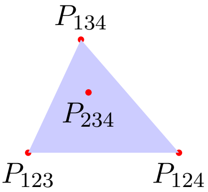

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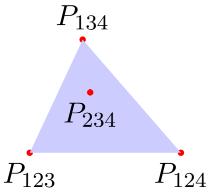

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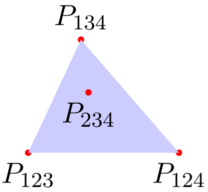

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$P_{234} \in X_1 \cap X_2 \cap X_3 \cap X_4$

Helly's theorem, special case, cont.

case (2): none of the P_I's lies in the convex hulls of the others

Helly's theorem, special case, cont.

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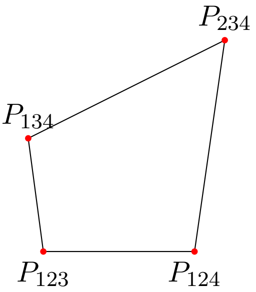
P_{123}, P_{134}, P_{124}, P_{234} are vertices of a convex quadrilateral



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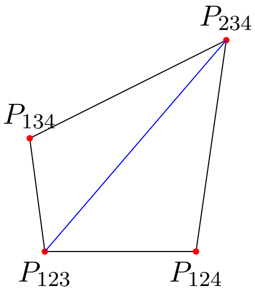

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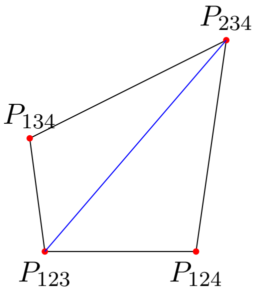

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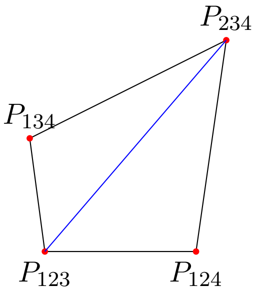

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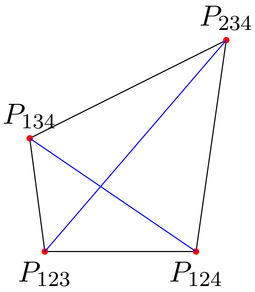

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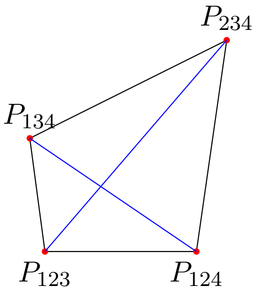

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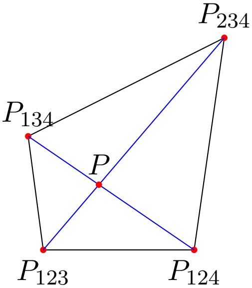

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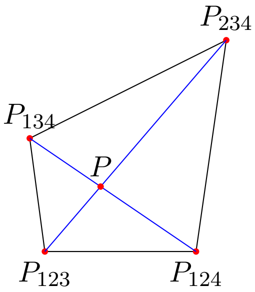

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by induction $Y_1 \cap Y_2 \cap \cdots \cap Y_n \neq \emptyset$
but $Y_1 \cap Y_2 \cap \cdots \cap Y_n = X_1 \cap X_2 \cap \cdots \cap X_n \cap X_{n+1}$

Intersections of infinite families of figures

Example 1.

$$X_n \;=\; \left\{ x \in \mathbb{R} \;:\; 0 < x < \frac{1}{n} \right\} \;=\; \left(\frac{1}{n}, 1 \right)$$

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Helly's theorem for infinite families

Fact. If $\{F_i\}_{i\in I}$ is a family of compact figures and $\bigcap_{i\in I} F_i = \emptyset$ then there exist indices $i_1, i_2, \ldots, i_n \in I$ such that $\bigcap_{k=1}^{n} F_{i_k} = \emptyset$.

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Let X_i, $i \in I$, be a collection of convex **compact** figures. If the intersection of every 3 of these figures is nonempty, then the whole collection has a nonempty intersection.

Perpendiculars to sides of a polygon

Let P be a polygon. Suppose that for each set of 3 sides of P there exists at least one point A from which one can draw 3 perpendicular segments respectively to these given sides (the point A may vary with the choice of the sides). Then there exists at least one point B such that for each side of P a perpendicular segment from B to that side can be drawn.

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Convex version:
Let P be a convex polygon. Suppose that for each set of 3 sides of P there exists at least one point $A \in P$ from which one can draw 3 perpendicular segments respectively to these given sides (the point A may vary with the choice of the sides). Then there exists at least one point $B \in P$ such that for each side of P a perpendicular segment from B to that side can be drawn.

Covering disc

Let $P_1, P_2, \ldots, P_n$ be points of the plane such that $|P_iP_j| \leq 1$ for any i, j. There exists a point P such that a disc with center P and radius $\frac{1}{\sqrt{3}}$ containing all P_i's.

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Take X_i, X_j, X_k and assume that $|P_iP_j| \geq |P_iP_k|, |P_jP_k|$
Denote $\alpha = \sphericalangle P_iP_kP_j$

Covering disc, cont.

- case 1: $\alpha \geq 90^\circ$

Covering disc, cont.

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  let P be the middle of the segment $[P_i, P_j]$

$$P_i, P_j, P_k \in D\left(P, \frac{1}{2}|P_iP_j|\right) \subset \left(P, \frac{1}{2}\right) \subset D\left(P, \frac{1}{\sqrt{3}}\right).$$

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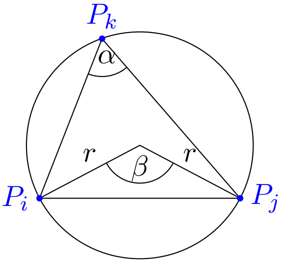
- case 2: $90^\circ > \alpha \geq 60^\circ$

$$\beta = 2\alpha$$



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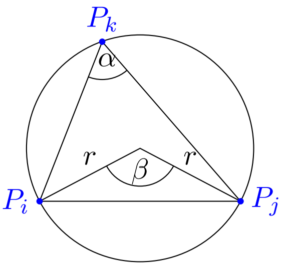

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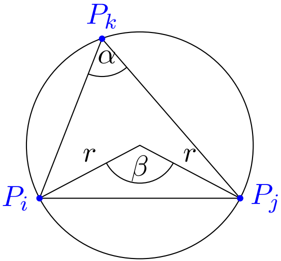

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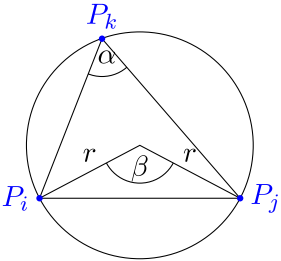

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Idea:
denote $P = \{P_1, P_2, \ldots, P_n\}$

$$\mathcal{F} \ = \ \left\{\pi \ : \ \pi \text{ is a half-plane, } |P \cap \pi| > \frac{2n}{3}\right\}$$

Convex body is a compact convex subset of plane with non-empty interior.

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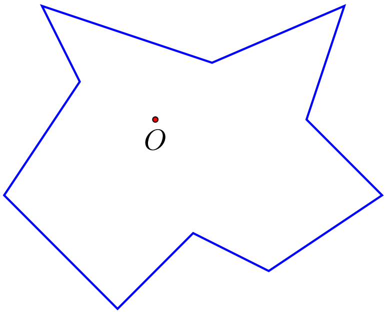
Let F be a convex body with the area S. There exists a point O such that for any closed half-plane π with border line passing through O

$$\text{area}(\pi \cap F) \ \geq \ \frac{1}{3}S.$$

Star-shaped polygons or art gallery problem



Polygon F (more generally, figure F) is **star–shaped** if there exists a point $O \in F$ such that for each point $P \in F$ the segment OP lies entirely within F.

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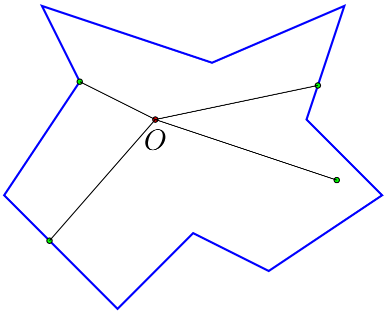

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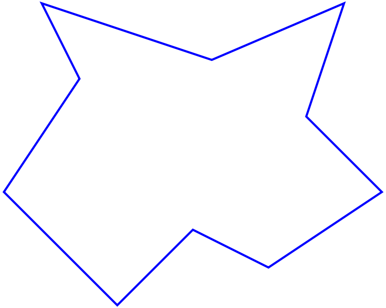
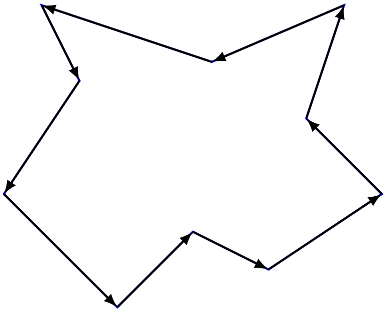
otherwords:
if there exists a point O from which any point of F is visible.

Star-shaped polygons or art gallery problem, cont.



Let F be a polygon. Assume that for any three points $P, Q, R \in F$ there exists a point O from which each of the points is visible. Then the figure is star-shaped.

Star-shaped polygons or art gallery problem, cont.



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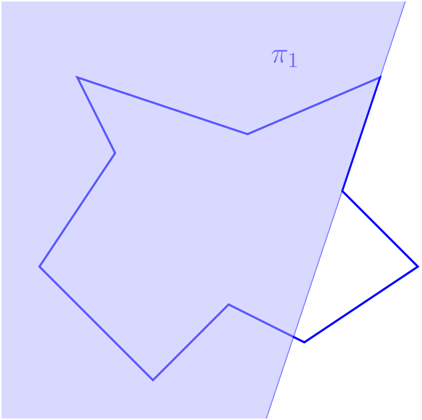

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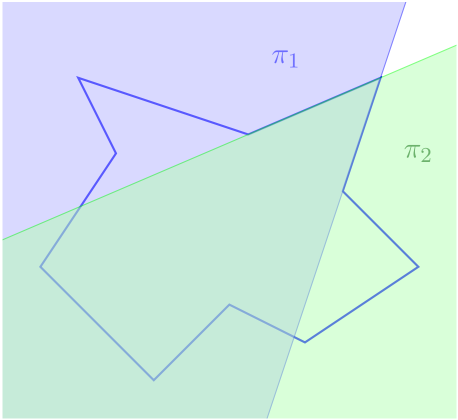

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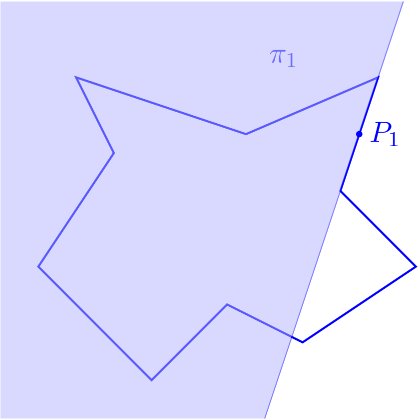

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- if the middle point P_i of a side is visible from a point O then $O \in \pi_i$;

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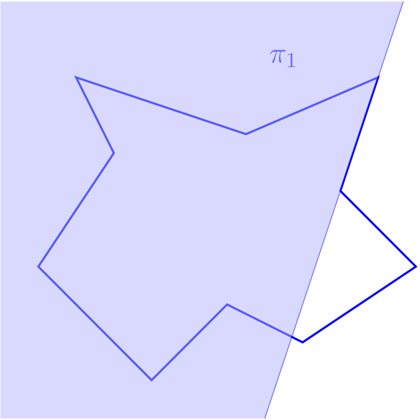

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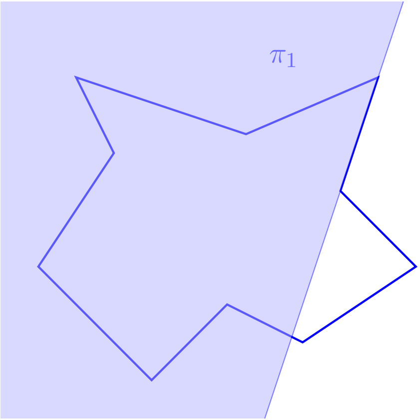

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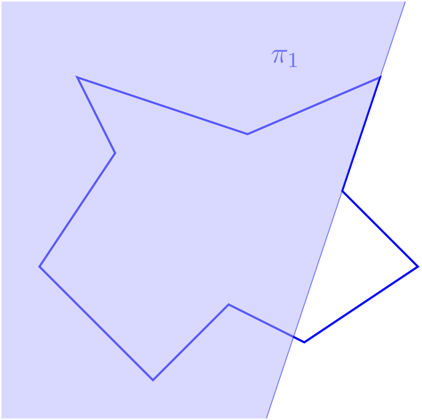

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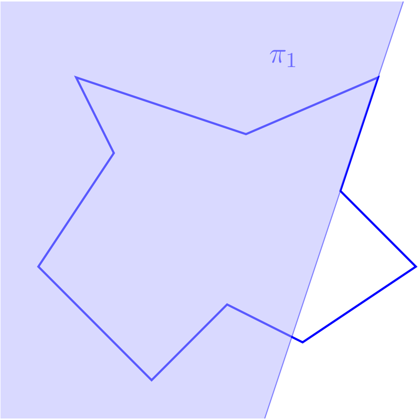

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Transversal to a family of figures

Let $\mathcal{F} = \{F_i\}_{i \in I}$ be a family of figures.
Transversal to $\mathcal{F}$ is a line ℓ that meets every figure in the family:

$$\ell \cap F_i \neq \emptyset \text{ for each } i \in I.$$

Transversal to a family of parallel segments

Let $\mathcal{F} = \{I_1, I_2, \ldots, I_n\}$ be a family of parallel segments s.t. there exists a transversal to every three of them. There exists a transversal to all of them.

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$$F_j \; = \; \{\, (a, b) \in \mathbb{R}^2 \mid (-x_j)a + y_j' \; \leq \; b \; \leq \; (-x_j)a + y_j'' \,\}$$

Transversal, cont.

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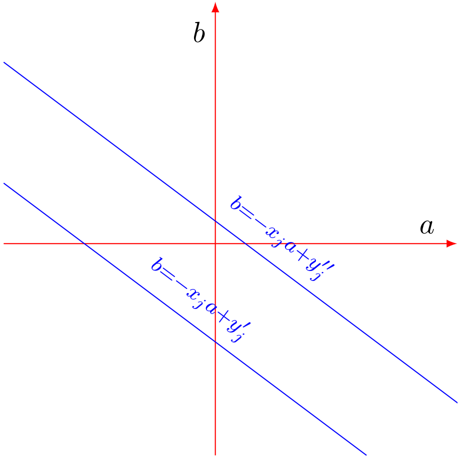

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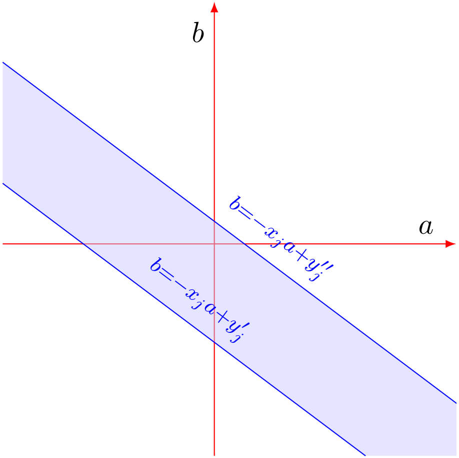

Klee's theorem

Notation: S any subset of the plane, v a vector
$T_v(S)$ - translation of S by the vector V

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Let S be a convex body and $\mathcal{F}$ a family of compact subsets in the plane. Assume that for any 3 sets in $\mathcal{F}$ there is a vector v such that $T_v(S)$ covers these 3 sets. Then there is a vector v_0 such that $T_{v_0}(S)$ covers all the sets in $\mathcal{F}$.

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V_S is

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Notation: S any subset of the plane, v a vector
$T_v(S)$ - translation of S by the vector V

Let S be a convex body and $\mathcal{F}$ a family of compact subsets in the plane. Assume that for any 3 sets in $\mathcal{F}$ there is a vector v such that $T_v(S)$ covers these 3 sets. Then there is a vector v_0 such that $T_{v_0}(S)$ covers all the sets in $\mathcal{F}$.

For $F \in \mathcal{F}$, lets denote

$$V_F = \{ v \ : \ S \subset T_v(F) \}$$

V_S is

- convex;
- bounded;
- closed.

Klee's theorem variants

exchange relation *contains* to

- intersects
  Assume that for any 3 sets in $\mathcal{F}$ there is a vector v such that $T_v(S)$ intersects these 3 sets. Then there is a vector v_0 such that $T_{v_0}(S)$ intersects all the sets in $\mathcal{F}$.

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- is included in
  Assume that for any 3 sets in $\mathcal{F}$ there is a vector v such that $T_v(S)$ is included in these 3 sets. Then there is a vector v_0 such that $T_{v_0}(S)$ covers all the sets in $\mathcal{F}$.

Covering of mirror image

Assume that
F is a convex body
F' is the image of F in a central symmetry
$2F$ is "doubling" of F

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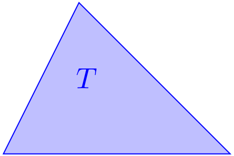
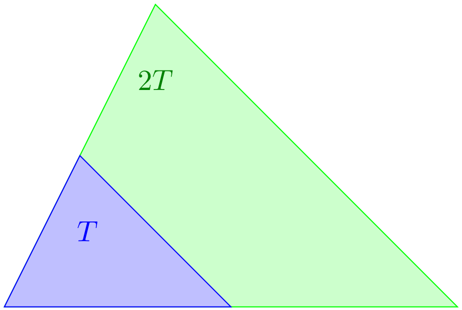
There exists a vector v such that $T_v(F)$ covers F'.

Special case of a triangle



Special case of a triangle



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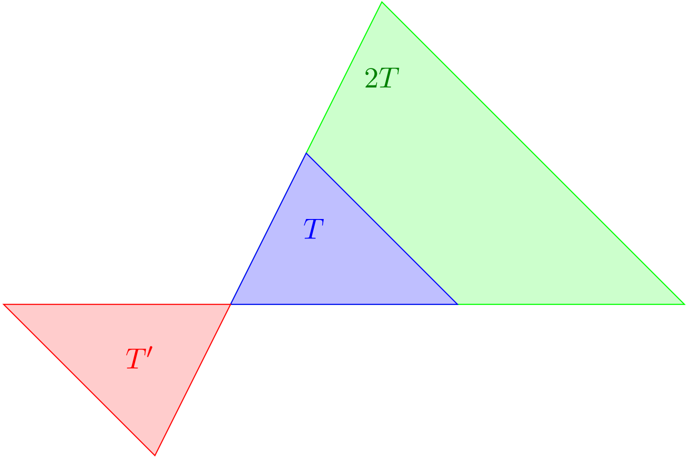
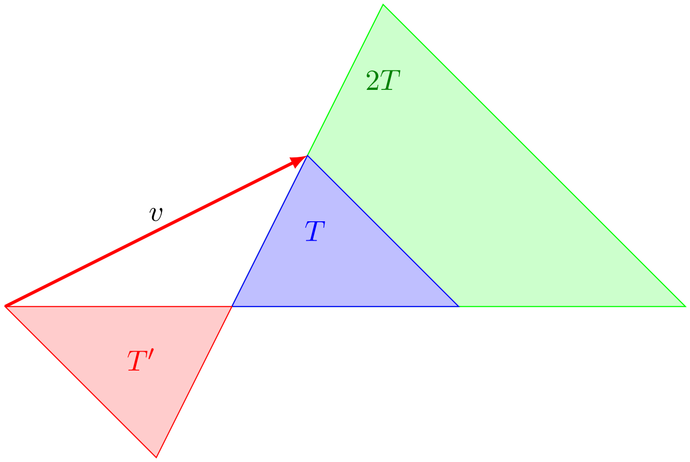


Special case of a triangle



Special case of a triangle