Decision problems and some mathematical games

Invariants in mathematical games

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Example — Negotiations

Two neighbors — Alice and Bob — negotiate the perfect place for the well between their houses.

- Alice proposes the placement
- Bob accepts it or proposes better one
- If Bob rejected, Alice accept new placement or propose another
- And so on

$c > 0$ — cost of one round of negotiations

How Alice should start the negotiations?

Game types

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3. Real life

   Difficulties:

   - Comparison of results (proper model)
   - Incomplete information
   - Random events
   - Irrational players

Invariants

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Example

Object: set of cars on a car-park. Invariants:

- number of wheels is divisible by 4
- number of doors (excluding trunks) is even
- number of mirrors is divisible by 3
- more?

Example

Set of n integers, for some natural n, is given. In one step we substitute 2 chosen numbers — a and b — by their difference $a - b$. After $n - 1$ steps, we have exactly one number.

- Give an example of 5 integers, for which it is possible to obtain 0. How to do it? Is it possible to obtain another number?

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- Give an example of 5 integers, for which it is impossible to obtain 0. Justify your answer.

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- Give an example of 5 integers, for which it is possible to obtain 0. How to do it? Is it possible to obtain another number?
- Give an example of 5 integers, for which it is impossible to obtain 0. Justify your answer.
- Is it possible for numbers: $1, 2, 3, \ldots, n - 1, n$?

Problem 0

Is it possible to go by chess knight from one corner of the chessboard to the opposite one, visiting every cell exactly once?

Problem 1

Numbers $a_1, a_2, \ldots, a_n$ are given. Assume that $a_j = 1$ or $a_j = -1$ for each $j = 1, 2, \ldots, n$, and

$$a_1a_2a_3a_4 + a_2a_3a_4a_5 + \ldots + a_{n-1}a_na_1a_2 + a_na_1a_2a_3 = 0.$$

Prove that $4|n$.

Problem 2

Numbers 1, 2, 3, ..., 2024 are written on the board. In one move we erase two chosen numbers a and b and write $\sqrt{a^2 + b^2}$ instead of them. Find the number left after 2023 moves.

Problem 3

Numbers $1, 2, 4, 5$ are written on the board. In one step we can replace two chosen numbers a and b by numbers $\frac{a+b}{\sqrt{2}}$ and $\frac{a-b}{\sqrt{2}}$. How many steps do we need to obtain the sequence $1, 2\sqrt{2}, 3, 4\sqrt{2}$?

Problem 4

Numbers $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{2023}, \frac{1}{2024}$ are written on the board. We choose two numbers — a and b — erase them and write number $a + b + ab$. We repeat this procedure 2023 times. What numbers could be obtained in the end?

Problem 5

Number $(2024!)^{2024!}$ is given. We sum up its digits, and repeat this procedure up to getting a single digit number. Find this number.

Semi–invariants

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Example

Set of n positive integers, for some natural n, is given. In one step we substitute 2 chosen numbers — a and b — by their greatest common divisor $\gcd(a, b)$.

Semi-invariant — sum of all numbers

Problem 6

There are n natural numbers written along a circle. Between each two neighboring numbers we write their gcd and erase original numbers. Prove that after some finite number of iterations of this procedure, all numbers will be equal.

Problem 7

In every cell of a chessboard some real number is written. In one step we can multiply by -1 all numbers in a chosen row or column. Is it possible to bring out the situation, in which sums of numbers in every row and column is nonnegative? Justify your answer.

Problem 8

There are n numbers written on the board. In one move we erase two chosen numbers a and b and write $\frac{a+b}{4}$ instead of them. Prove that, if we start with n ones, then after $n-1$ moves we get a number not lesser than $\frac{1}{n}$.

Problem 9

There are 100 knights in king Arthur's court. Each of them has at most 49 enemies. Is there exist a sitting at the round table such that enemies are not neighbors?