Decision problems and some mathematical games

Smart players

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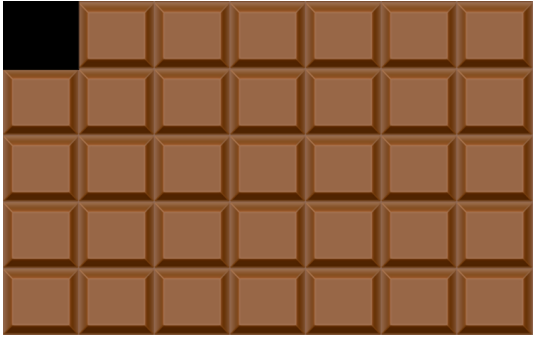
Liar twins

Twin brothers are protecting the treasure. One always says true, second is a permanent liar. You meet one of them at the fork of the road. You know that one way leads to the treasure, but the second — to the abyss. When you choose one, can't go back.
What question (only one) would you ask the met twin, to figure out where the treasure is?

Poisoned chocolate

The top left corner of the chocolate bar is poisoned. Two chocolate lovers have to divide the bar between themselves. They can break — alternately — all stripes (arbitrary wide) from the right or bottom. Find the smart strategy for both of them. Which of them can avoid poisoning?



Stones

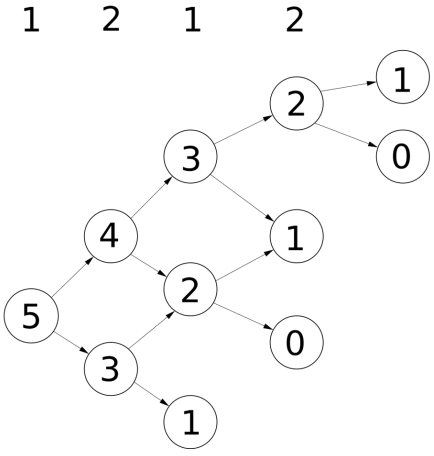
There is a pile of N stones on a table. There are 2 players who get — alternately — 1 or 2 stones from the pile. The game is over when the table is empty, and the one who get the last stone is a looser.
Which player has a winning strategy? Find this strategy.

Stones — upgrade

There is a pile of N stones on a table. There are 2 players who get — alternately — 1 or 2, or 3, ..., or M stones from the pile (where $M \leq N$). The game is over when the table is empty, and the one who get the last stone is a looser.
Which player has a winning strategy? Find this strategy.

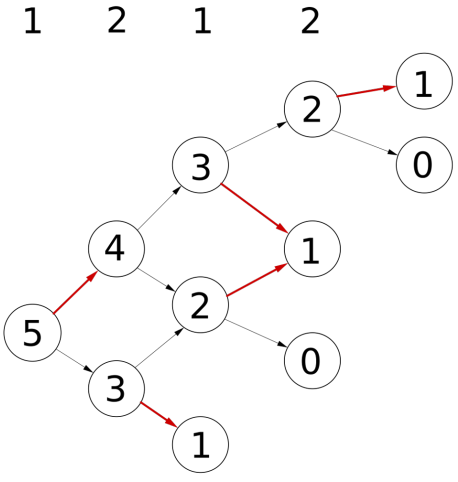
Example

5 stones



Example

5 stones — smart strategies



Extensive games

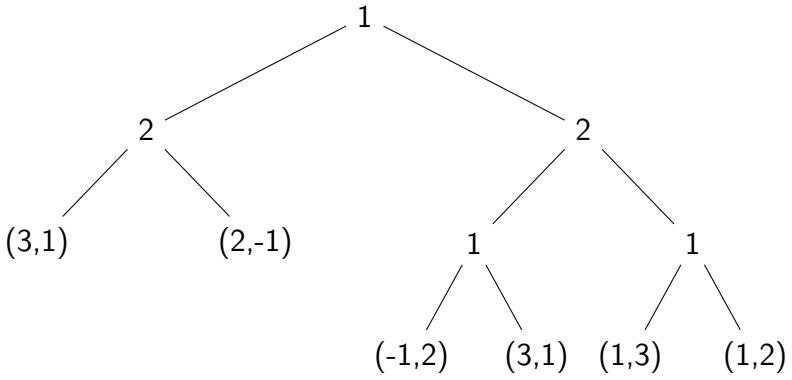
Extensive game

Pair: set of players, and set of all possible courses of the game (histories).

Remark

It is convenient to express the extensive game as a mathematical three.

Find the smart strategies



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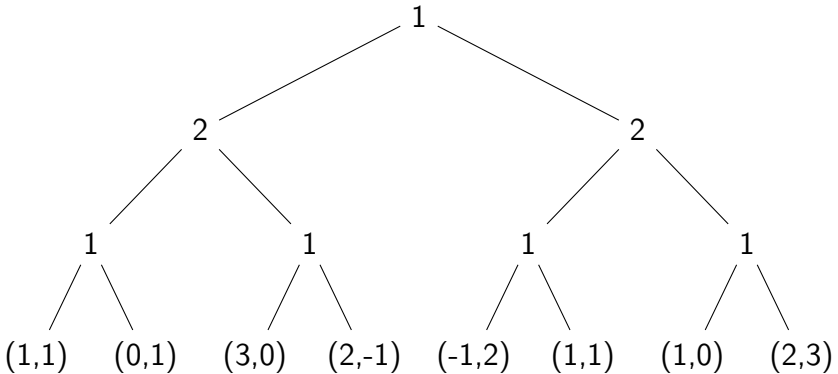

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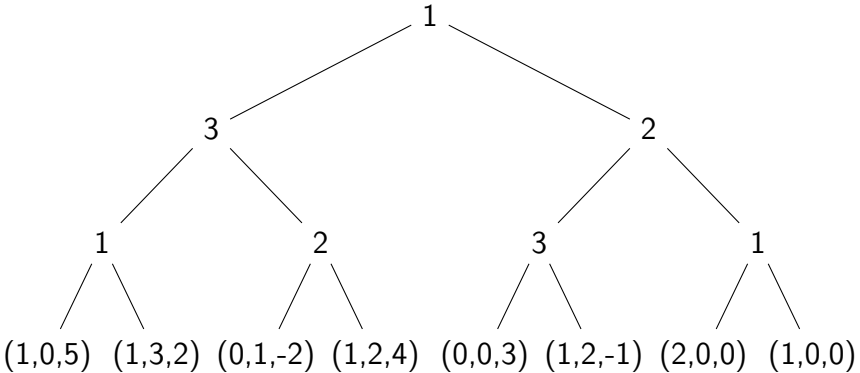

Conclusions and remarks

1. Smart strategy can be ambiguous.
2. If other players play smart, I need to be smart too.
3. System of smart strategies is called Nash equilibrium.

Nash equilibrium

System of strategies such that changing of strategy by one player gives him/her lesser profit.

Strategic game

Strategic game

Set of players, and sets of all possible strategies of all players.

Remark

It is convenient to express the strategic game for 2 players as a table.

Nash equilibrium in strategic games

Example — production planning

There are N producers of some good on the market. Denote

- s_i — production volume,
- k_i — costs of production,
- c — prize,
- $u_i(s_1, \ldots, s_N) = s_i c(s_1 + \ldots + s_N) - k_i(s_i)$.

What is the optimal production volume of each producer?

Nash equilibrium in strategic games

Production planning, $N = 2$

- $u_i(s_1, s_2) = s_i c(s_1 + s_2) - k_i(s_i)$

Nash equilibrium in strategic games

Production planning, $N = 2$

- $u_i(s_1, s_2) = s_i c(s_1 + s_2) - k_i(s_i)$
- Strategic game

Strategy	l_2	m_2
l_1	(0,2)	(2,3)
m_1	(1,1)	(1,0)

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	$\downarrow$		$\uparrow$
m_1	**(1,1)**	$\leftarrow$	(1,0)

- System of strategies (l_1, m_2) and (m_1, l_2) are optimal (they are Nash equilibria).

Find all Nash equilibria

Strategy	a_2	b_2	c_2
a_1	(8,3)	(6,4)	(15,0)
b_1	(9,1)	(5,2)	(6,3)
c_1	(3,2)	(4,3)	(5,4)
d_1	(2,66)	(3,10)	(4,8)

Find all Nash equilibria

Strategy	a_2	b_2	c_2
a_1	(8,3)	(6,4)	(15,0)
b_1	(9,1)	(7,2)	(6,3)
c_1	(3,2)	(4,3)	(5,4)
d_1	(2,66)	(3,10)	(4,8)

Find all Nash equilibria

a_3	a_2	b_2	c_2
a_1	(2,0,1)	(2,0,1)	(2,0,1)
b_1	(2,0,1)	(1,2,0)	(0,1,2)
c_1	(2,0,1)	(0,1,2)	(0,1,2)

b_3	a_2	b_2	c_2
a_1	(2,0,1)	(1,2,0)	(0,1,2)
b_1	(1,2,0)	(1,2,0)	(1,2,0)
c_1	(2,0,1)	(1,2,0)	(0,1,2)

c_3	a_2	b_2	c_2
a_1	(2,0,1)	(0,1,2)	(0,1,2)
b_1	(0,1,2)	(1,2,0)	(0,1,2)
c_1	(0,1,2)	(0,1,2)	(0,1,2)