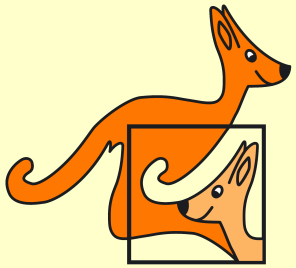
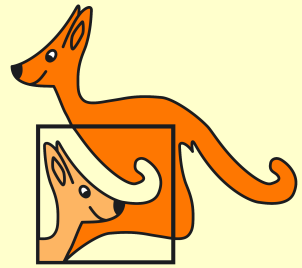
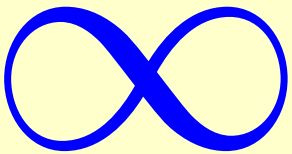


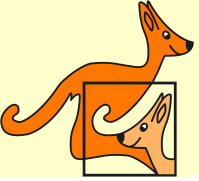
Zakopane 2024



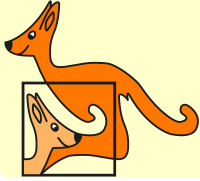
Conquest of infinity



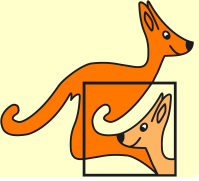
Mieczysław K. Mentzen



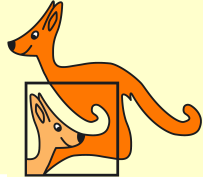
Part I



A little bit of history



Introduction



The mysterious concept of infinity in various forms has accompanied philosophers for millennia. It was usually understood intuitively as something unimaginably large, vast and incomprehensible. We owe a certain systematization of this concept to the ancient Greeks.

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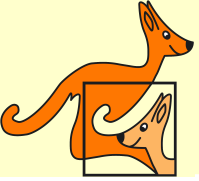
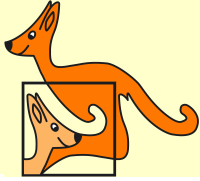
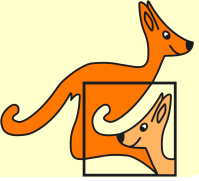


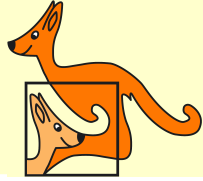
Two kinds of finity



The Greeks distinguished two meanings that could be hidden under the term *infinity*. Namely, the Greeks introduced terms that today can be expressed as *potential infinity* and *actual infinity* (*completed infinity*).



Potential infinity



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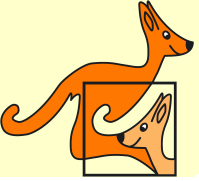

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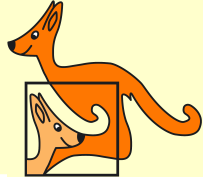

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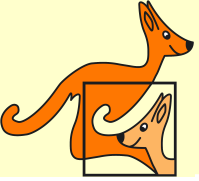
An example of potential infinity can be natural numbers written on a piece of paper. Regardless of how many such numbers we write (how many we think about), we can always add one more. If the reader makes an effort to write down all the natural numbers from 1 to 1000, he can add 1001, and then 1002, etc.



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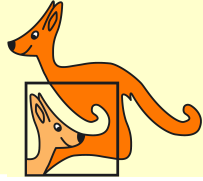
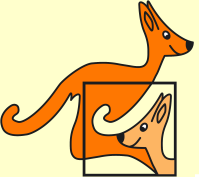


Another (but similar) example of such infinity is the division of a line segment into smaller segments by points. Regardless of how many parts we divide such an segment into, it can always be divided into more parts.



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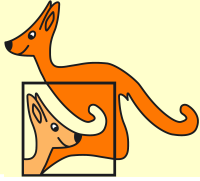

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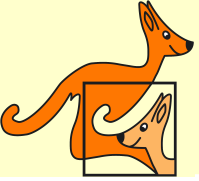
A segment divided into 14 parts.



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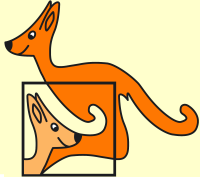

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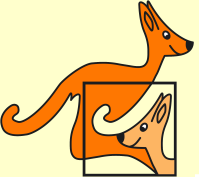
A segment divided into 14 parts.

The same segment divided into 17 parts.



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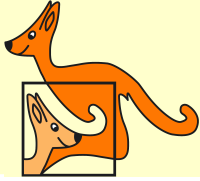
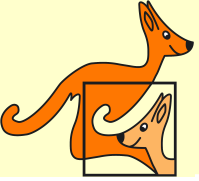


A modern example of a situation related to potential infinity is the concept of a *limit*. We will not focus on defining this concept, we will content ourselves with intuitive understanding.



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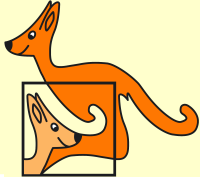
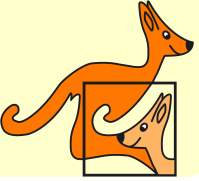


If for instance x goes to 3, then x^2 goes to 9. Popular notations used for the limit are

$$\lim_{x \to 3} x^2 = 9 \qquad \text{or} \qquad x^2 \xrightarrow[x \to 3]{} 9.$$

We use the terminology: x *tends* to 3, x^2 *tends* to 9. That means: x changes its value to be closer and closer to 3, then x^2 also changes its value to be closer to 9.



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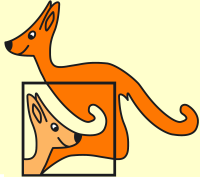

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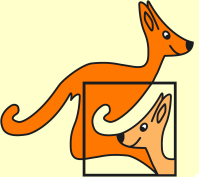
Consider the example of a limit:

$$\lim_{x\to\infty} x \sin \tfrac{1}{x} = 1.$$

Here x gets larger and larger, however it never be equal to infinity. We may say that x "is potentially infinite". This is just the modern understanding of potential infinity. In the above example the value of limit is 1, nevertheless $x \sin \frac{1}{x}$ is never equal to one, it is 1 only "potentially". We will note that currently mathematicians do not use the phrase "potential infinity" in such a context, they only use the concept of a limit.



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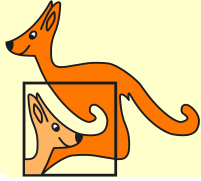

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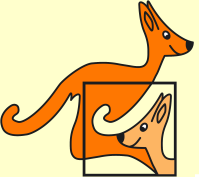

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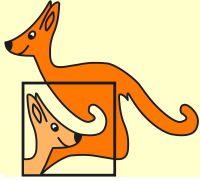

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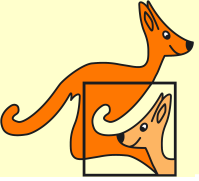
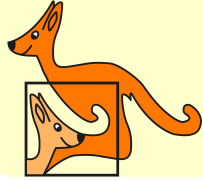
The name *limes* appeared in mathematics in the 17th century. It was used (independently) by Newton and Leibniz.

Newton alluded to continuous motion: at each particular point in time an object has some speed.

Leibniz explained the limit using the example of curves, when one curve can be infinitely close to another, but will not coincide with it.



Actual infinity

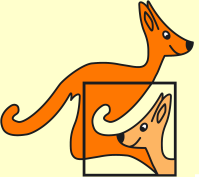


What is actual infinity?

This meant a quantity that is greater than any finite quantity. In the language of set theory, it can be expressed as follows: a set is infinite if each finite subset of it is only a part of the whole.

In other words, an infinite set contains infinitely many elements.
A modern example would be the set of all natural numbers:

$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \ldots\}.$$



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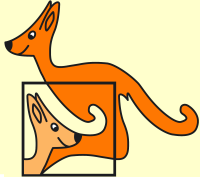

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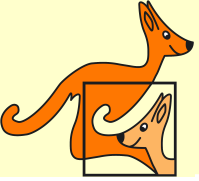
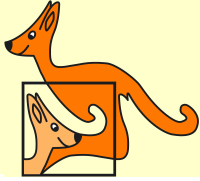
However, a doubt arises: whether such an infinite set as e.g. $\mathbb{N}$ really exists. And if so, where is it?

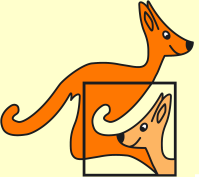
We will return to this issue later.



Actual infinity vs. potential infinity



What is the difference between these two types of infinity? Let's look at the examples given above.



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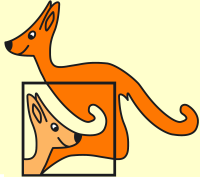

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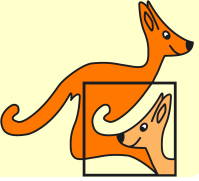
When we write natural numbers, we will always write only finitely many of them. Of course, we can add additional numbers (we have this option), but we will never write infinitely many numbers. The same applies to the division of the interval. At any given moment we get finitely many division lines, but there will never be an infinite number of them.



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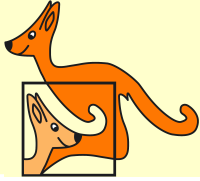

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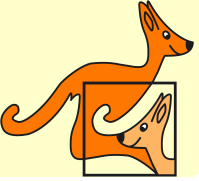
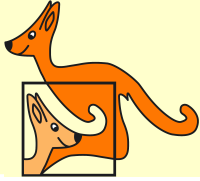
The situation is different with the set of all natural numbers. There really are an infinite number of them. Can you imagine an interval divided into infinitely many subintervals? Of course yes. After all, numbers $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$ divide the segment $\langle 0; 1\rangle$ into infinitely many segments. However, we cannot indicate all the natural numbers or mark infinitely many division points on the line.



How did the Greeks see it?



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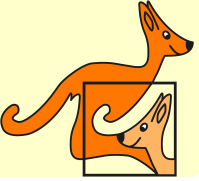

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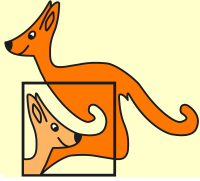

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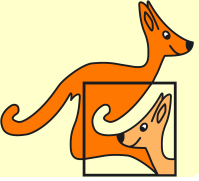

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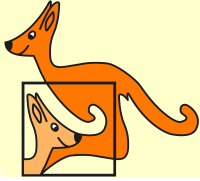

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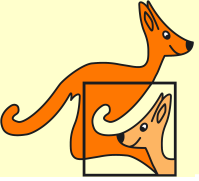
At the age of 17, Aristotle came to Athens and joined the famous Plato's Academy. He spent 20 years there — first as a student, then as Plato's assistant, and finally as an independent lecturer.



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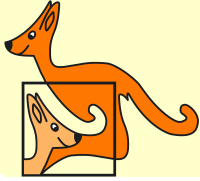

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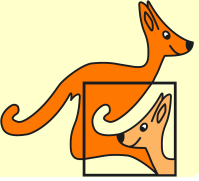

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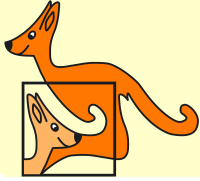

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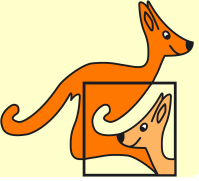

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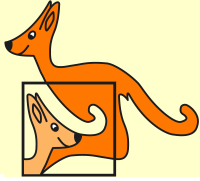

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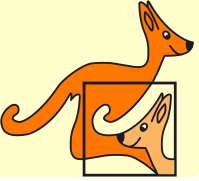
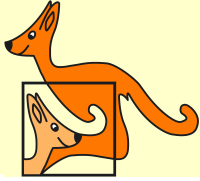
Aristotle is considered the father of natural sciences. Although many of his hypotheses turned out to be wrong[1], they contributed to the creation of new hypotheses and the development of natural sciences.

[1] For example, Aristotle thought that a spider had 6 legs. Another of his erroneous ideas was the belief that the heavier the objects fall, the faster they fall. This second hypothesis was falsified in 1586 by the Flemish scientist Simon Stevin.



The paradoxes of Zeno of Elea



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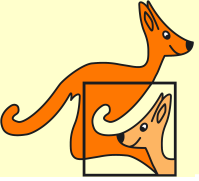

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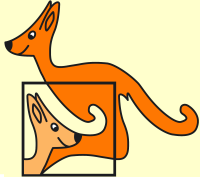

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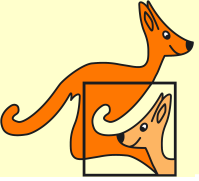

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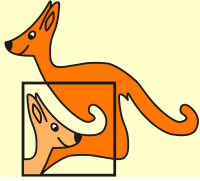

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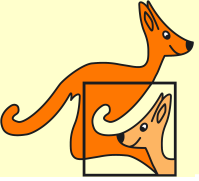

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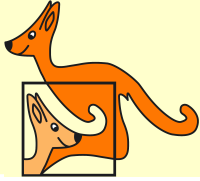

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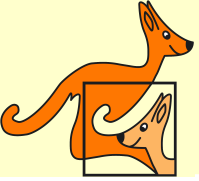
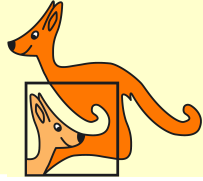
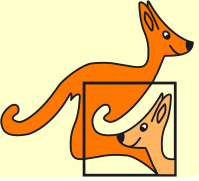
Below we will mention two of these paradoxes, commonly named *Dichotomy* and *Achilles and the Tortoise*. In both of them a vaguely imagined concept of infinity appears.



Dichotomy



The word "dichotomy" simply means division into two parts, separate and complementary to the whole.



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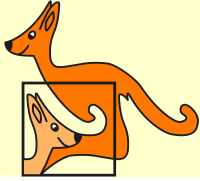

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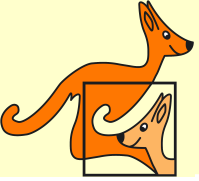

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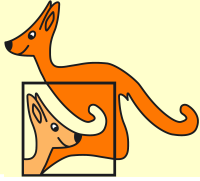

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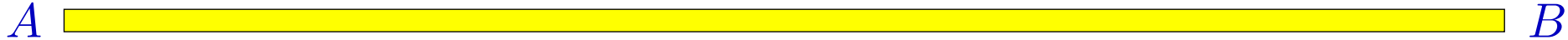

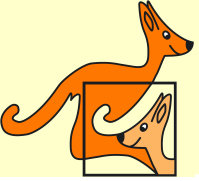

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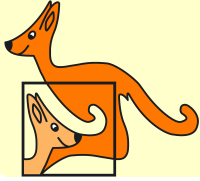

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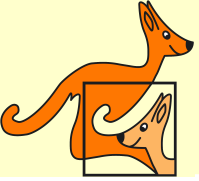

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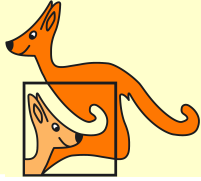

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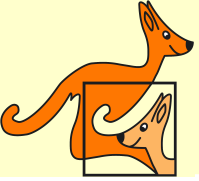

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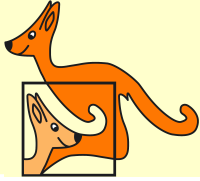

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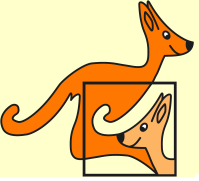

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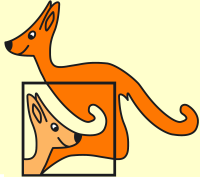

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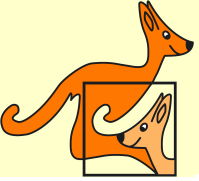

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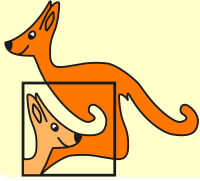

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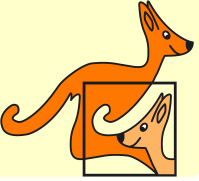
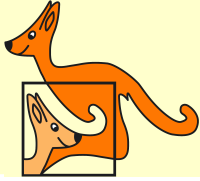
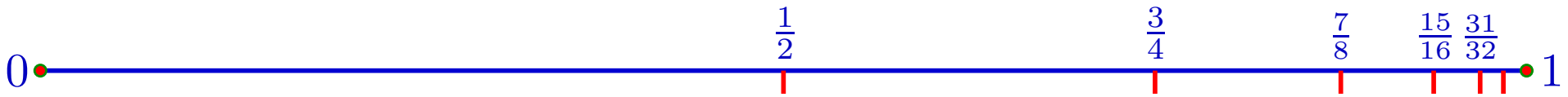
The listener is inclined to ask at this point whether this Zeno was not mad? After all, he saw objects, people and animals in motion. Yes, he saw it, but he considered this movement an illusion.

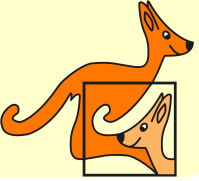


What is wrong in Dichotomy?

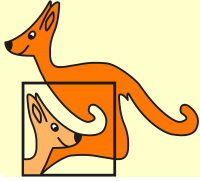


The situation described in Dichotomy is illustrated for $AB = 1$ in the following figure.





Achilles and the Tortoise



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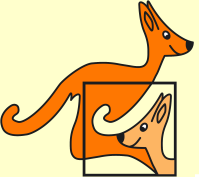

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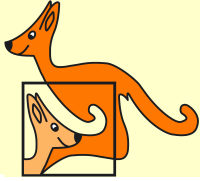

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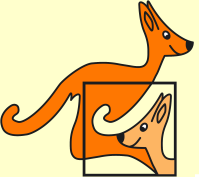

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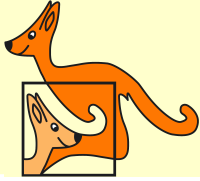

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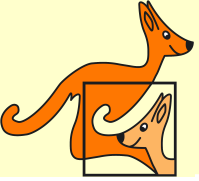
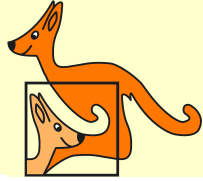
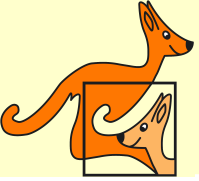
You guessed that this paradox is a more complicated version of the Dichotomy. However, Zeno's mistake is exactly the same as before. The time needed for Achilles to catch up with the turtle will be as much as it would take for Achilles to cover the initial distance from the turtle at a speed equal to the difference in the speed of the fast runner and the animal.



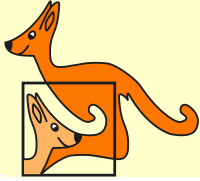
Be careful!



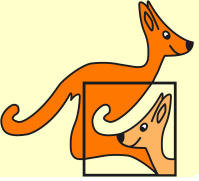
The examples presented above show that the world of infinity is nothing to joke about. Even stout heads were led astray in this mysterious world. It took many centuries for this issue to be approached from a different, appropriate angle. But we will talk about this in what follows.



Centuries pass...



The next dozen or so centuries did not bring any significant changes in the understanding of infinity — in fact, the belief was confirmed that there could be no actual infinity in our world. (Some theologians may have allowed such infinity in relation to God (almighty, infinitely merciful, ...), but God is a being from outside our world and impossible to comprehend by human reason).



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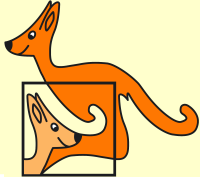
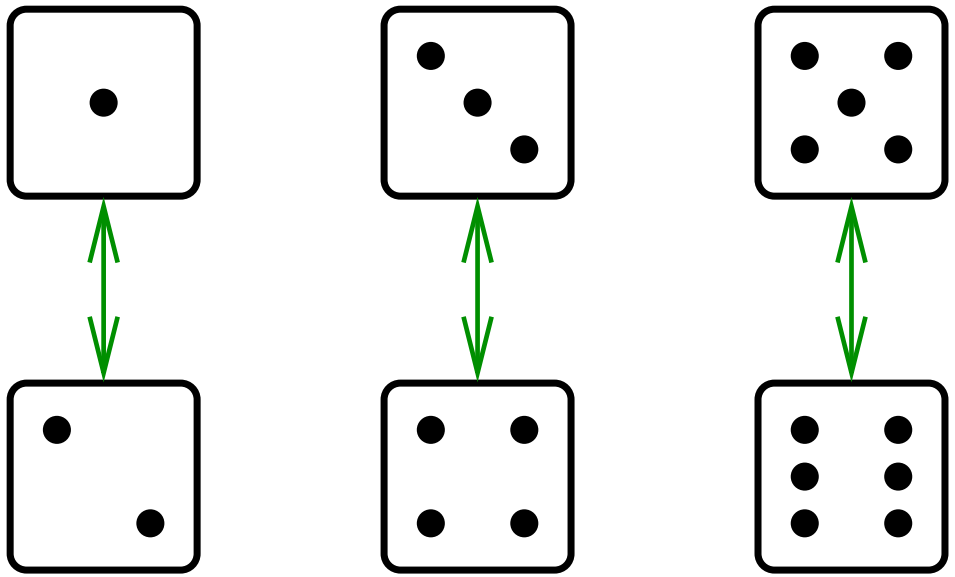
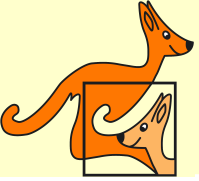


Arguments against the existence of actual infinity were based on the consequences of the observation that if the elements of two sets are combined in pairs (exhausting both sets), then these sets have the same number of elements (we now say that they are *equinumerous*). For example, the set of faces with an odd number of dice and the set of faces with an even number of dice of a six-sided dice are equal in number.



A dice has the same number of faces with an odd number of dots as faces with an even number of dots.



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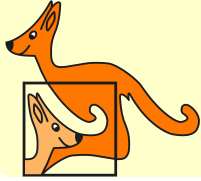
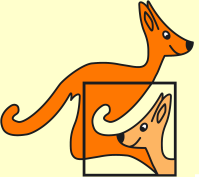


An example of an argument against the existence of infinity is an observation attributed to the grandmaster of self-promotion, Galileo.



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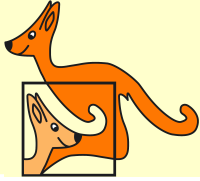

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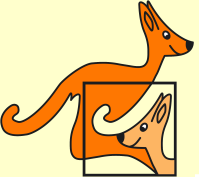
Galileo observed that, for example, natural numbers can be paired with even numbers:

$$(1;2),\ (2;4),\ (3;6),\ (4;8),\ (5;10),\ldots$$

both sets exhaustively (every even natural number can be written in the form $2n$, where n is a natural number).



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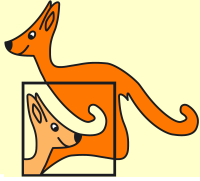

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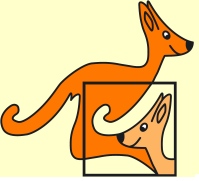

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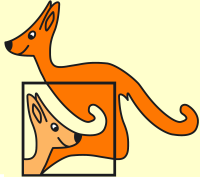

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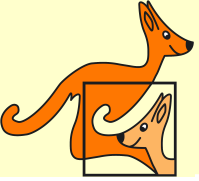
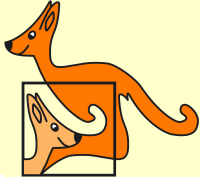
Several centuries have passed. At that time, many scientific authorities spoke out against actual infinity, and only a few tried to defend it. In such circumstances we reached the turn of the 18th and 19th centuries.



Bernard Bolzano



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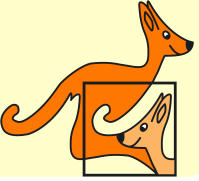
(5.10.1781 — 18.12.1848)

After graduating from the Piarist gymnasium, in 1796 Bernard junior began his studies at Charles University[2] in Prague: philosophical, mathematical and finally theological studies. After nine years, 7.04.1805, he was ordained a priest, then 17.06 of that year, a doctorate in philosophy, and two days later he took up the chair of theology at this university. He stayed in this position until 24.12.1819 — it was taken away because he came into conflict with the Austrian authorities.

[2]It is the oldest university in Central Europe, founded in 1348 by the Czech king Charles IV of Luxembourg.



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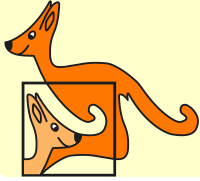

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Despite some problems with the authorities (today we would call them minor inconveniences), Bolzano wrote many articles on mathematics, including many groundbreaking ones. Now he had more time (no duties as a professor), he had his own source of income and many influential friends. Bolzan's scientific works were published only after his death. He died 18.12.1848 from tuberculosis, which was then incurable.

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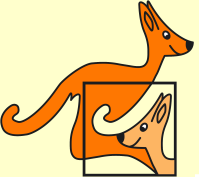

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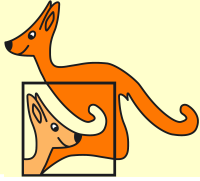
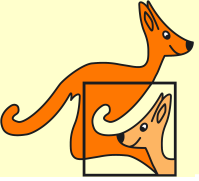


Bolzano defines actual infinity, which he called *true infinity*, as we did in the introduction: it is a quantity that is greater than any finite quantity. He gives the length of a straight line as an example. He emphasizes here that this is not a variable quantity (potential infinity).



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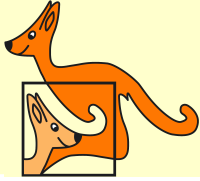

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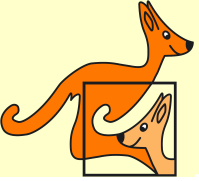

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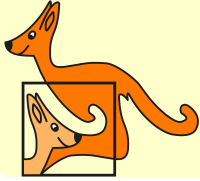

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Bolzano illustrated his considerations with many other examples of (actual) infinity, such as all points of a straight line or of a circle.

He also gave an interesting example of a logical nature, namely the set of all sentences.

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For example, let the sentence A_0 be: $2 + 2 = 4$. We now consider the sentence A_1 with the following content: *The sentence A_0 is true.* Of course, these sentences are different — the sentence A_0 talks about a certain equality of natural numbers, and the subject of the sentence A_1 is not this equality, but... the sentence A_0. Now we formulate the sentence A_2 as follows: *The sentence A_1 is true.* The subject of this sentence is the sentence A_1, which is different from the subject of the sentence A_1. Next, we formulate a sentence A_3 with the content: *The sentence A_2 is true*, and so on ad infinitum. We get an infinite number of sentences.

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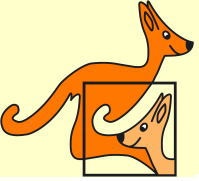
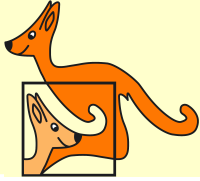
This last example also justifies that there are **really** infinitely many natural numbers, because there are the same number of them as there are sentences A_0, A_1, A_2, ...

Bolzano argues with some of his contemporary mathematicians who define infinity as a variable quantity (potential infinity) whose value increases indefinitely — it can be greater than any fixed value. He criticizes the escape of mathematicians from the concept of actual infinity into the vague concepts of *infinitely small quantities* and *infinitely great quantities*. Let us clarify here that an infinitesimal quantity is "almost zero", but not zero. An infinitesimal quantity is simply a kind of potential infinity — unlimited growth of real numbers[3].

[3]The concepts of *infinitely small* and *infinitely large* quantities have been replaced over time by the concept of *limit*. The limit replaced, among others, objects that corresponded to potential infinity understood as the possibility of unlimited growth.



Bernard Bolzano



As we mentioned earlier, Bolzan's scientific works were published only after his death. However, they had a significant impact on the attitude of mathematicians towards the concept of actual infinity. Currently, such infinity is as natural to mathematicians as, for example, the concept of a right angle. Modern mathematicians don't even use the phrase "actual infinity", because infinity is infinity. They freely operate with infinite sets: the set of all natural numbers, the set of all integers, the set of all real numbers, etc.

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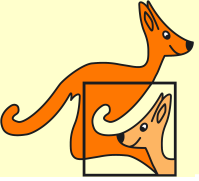
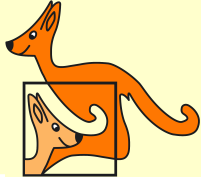
Bernard Placidus Johann Nepomuk Bolzano
(5.10.1781 — ∞)

End of part I