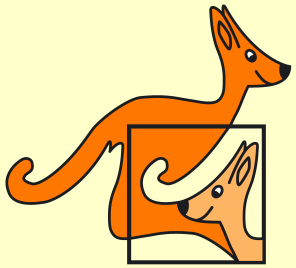
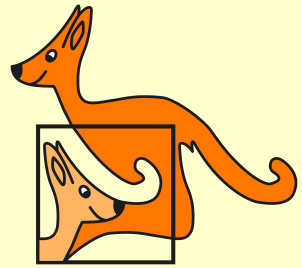
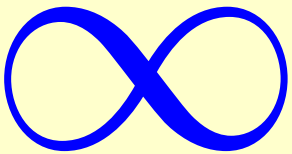


Zakopane 2024



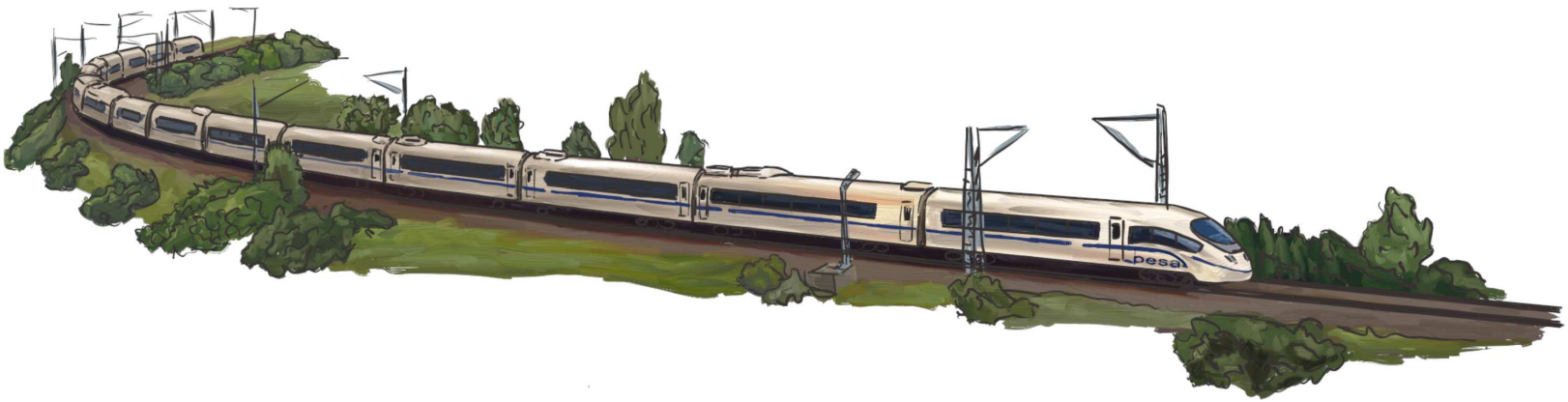
Conquest of infinity

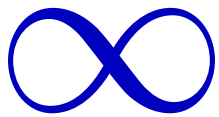


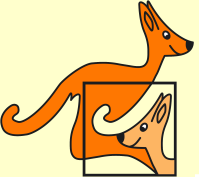
Mieczysław K. Mentzen

Part II

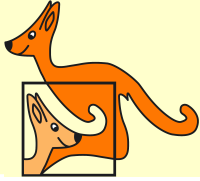
A journey to the Kingdom of Mathematics





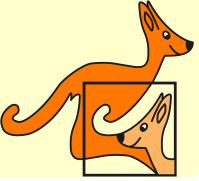


Infinite sets



What is an infinite set? We will formulate a definition of such a set borrowed from Dedekind[1]. However, we will start by recalling the concept of equinumerosity mentioned earlier.

[1] Julius Wilhelm Richard Dedekind, born 6.10.1831 in Brunswick, died 12.02.1916 — German mathematician. He was a student of Dirichlet and Gauss. From 1862, for over 50 years he was a professor at the Collegium Carolinum in Braunschweig. He was close friends with Cantor and was one of the first to appreciate the value of his set-theoretic works.



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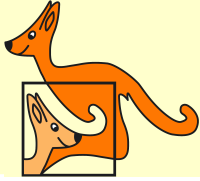

What is an infinite set? We will formulate a definition of such a set borrowed from Dedekind[1]. However, we will start by recalling the concept of equinumerosity mentioned earlier.

Definition. We say that two sets A and B are *equinumerous*, if the elements of these sets can be paired in such a way that each element of A is paired with exactly one element of the set B, and each element of the set B is paired with exactly one element of the set A.

In other words, sets A and B are *equinumerous* if there exists a one-to-one and onto correspondence between them. This means that it must exist a function from A to B such that for every element of B there exist just one corresponding element of A. (Such a function is called a *bijection*).

---

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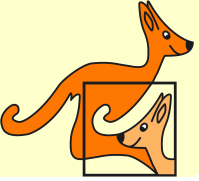

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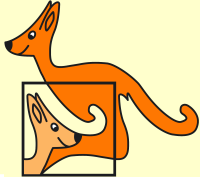

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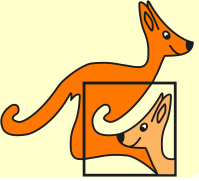

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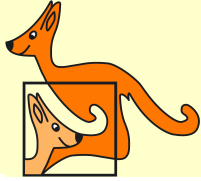
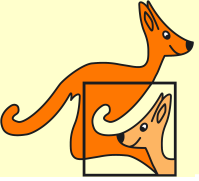


In the first example of equinumerous sets mentioned earlier, the elements of the set A were the faces of a six-sided dice with an odd number of dots, and the elements of the set B were the faces of this dice with an even number of dots.

The second example is the equinumerosity of the set of all natural numbers and the set of even natural numbers — the pairs there had the form $(2; 2n)$, $n = 1, 2, 3, \ldots$.
Here the bijection that establishes the equinumerosity has the form $f : \mathbb{N} \to 2\mathbb{N}$, $f(n) = 2n$.



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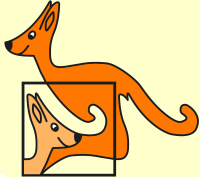
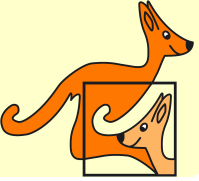


It's time to define an infinite set.

[2] A *proper* subset is a subset to which some elements of a given set do not belong (at least one does not).



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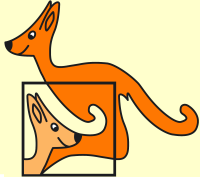

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Definition. A set A is called *infinite* if it is equinumerous with some its proper subset[2].

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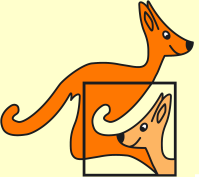
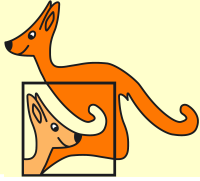


Infinite sets



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So we immediately have at least one infinite set — the set of all natural numbers — equinumerous with the set of all even numbers.

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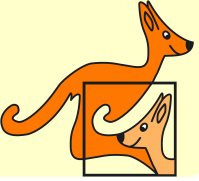

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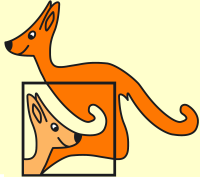

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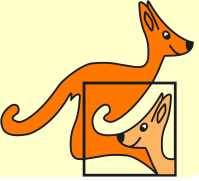
So we immediately have at least one infinite set — the set of all natural numbers — equinumerous with the set of all even numbers.

Let us denote the set of all natural numbers with the symbol $\mathbb{N}$. Let's establish that

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\},$$

i.e. we will consider natural numbers starting with zero.

[2] A *proper* subset is a subset to which some elements of a given set do not belong (at least one does not).



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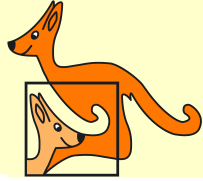
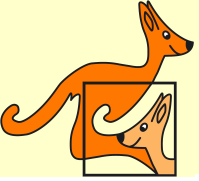


Example. The set of even natural numbers is infinite.

Indeed, pairs of the form $(2n; 4n)$, $n \in \mathbb{N}$, show that the set of even natural numbers is equinumerous with the set of natural numbers divisible by 4 (natural numbers divisible by 4 can be expressed as $4n$, where $n \in \mathbb{N}$). The set of natural numbers divisible by 4 is a proper subset of the set of even numbers. Therefore, the set of even numbers is equinumerous with its proper subset, and therefore it is infinite.



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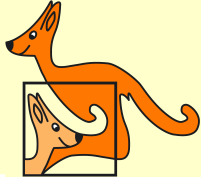

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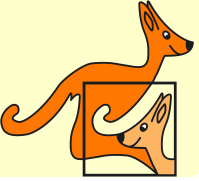

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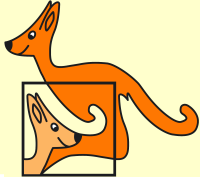

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Solution. Let $f : A \to B$ and $g : B \to C$ be bijections. Define $h : A \to C$ by $h(a) = g(f(a))$, $a \in A$.

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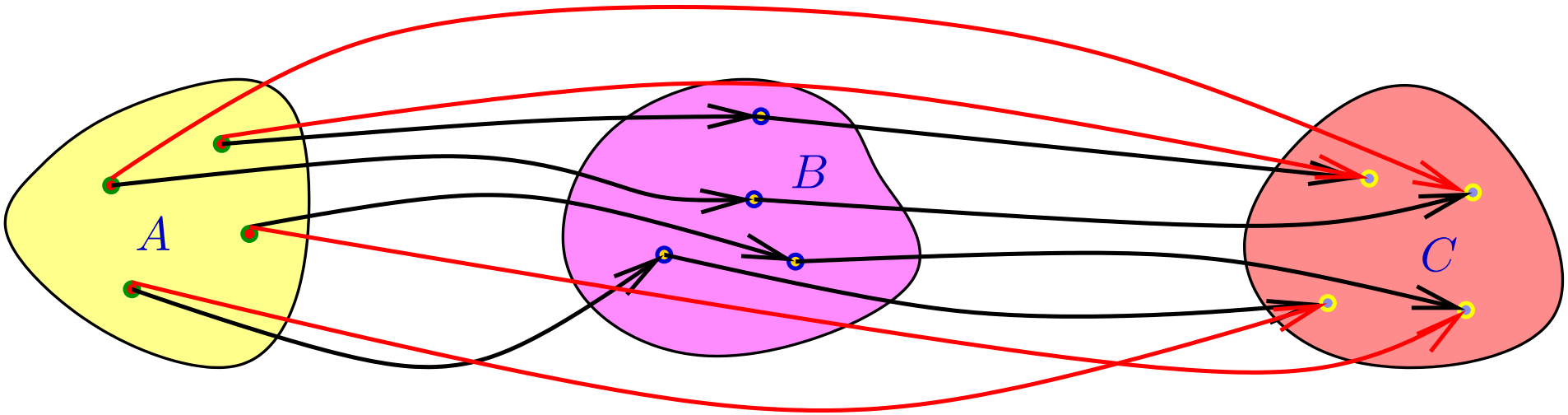

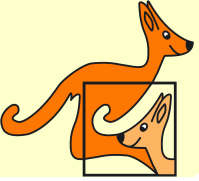

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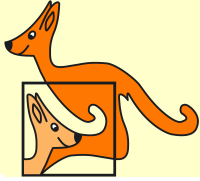

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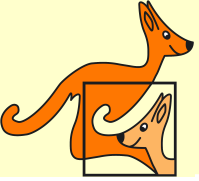

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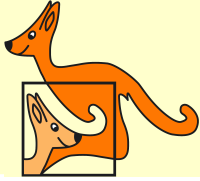

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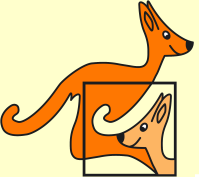

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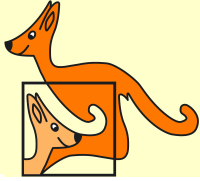

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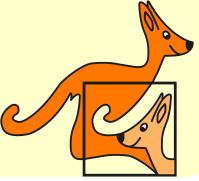
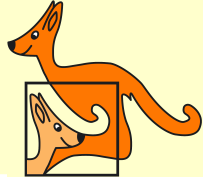
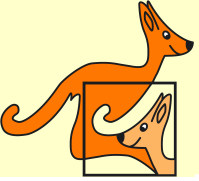
The function $h : A \to C$ turned out to be a bijection, so the sets A and C are equinumerous. $\square$



Countable sets



Definition. Set equinumerous with the set of natural numbers is called *countable*.



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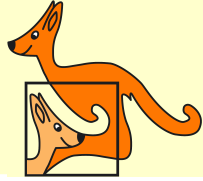

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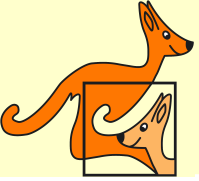

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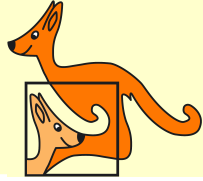

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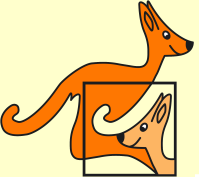

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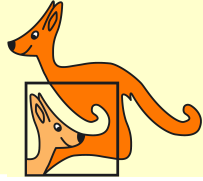

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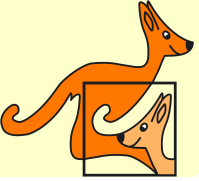

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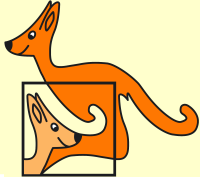

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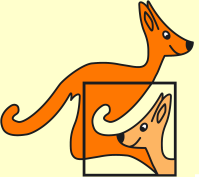
Hint. If $f : \mathbb{N} \to A$ and $g : \mathbb{N} \to B$ are bijections, then lconsider the function

$$h(n) = \begin{cases} f(\frac{n}{2}) & \text{if } n \text{ is even,} \\ g(\frac{n-1}{2}) & \text{if } n \text{ is odd.} \end{cases}$$



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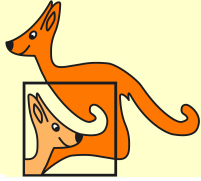

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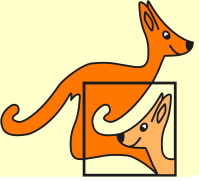

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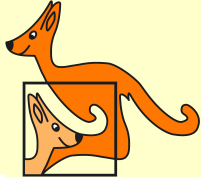

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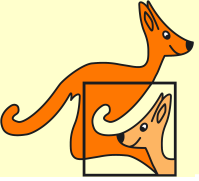
In such a way we have obtained a function $f : \mathbb{N} \to A$, $f(n) = a_n$. This function is clearly a bijection. $\square$



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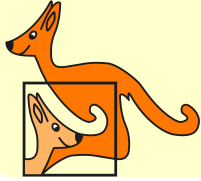

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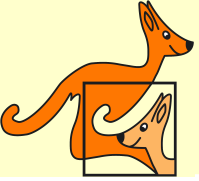

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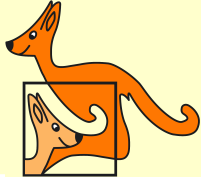

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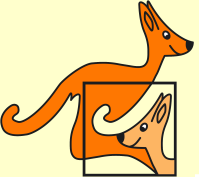

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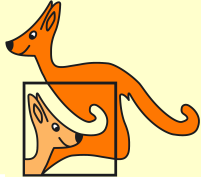

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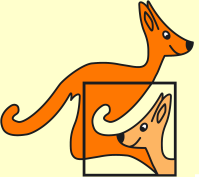
Solution. Let $N_0 = \{n \in \mathbb{N} :$ there is an $x \in A$ such that $f(x) = n\}$, i.e. N_0 is the image of A via f. Observe, that N_0 is equinumerous with A, so N_0 is infinite. By Problem 2, N_0 is countable, and so is A. □



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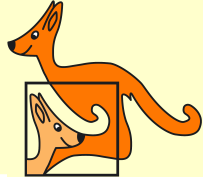

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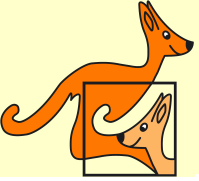

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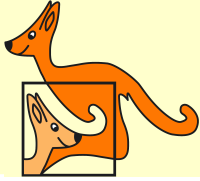

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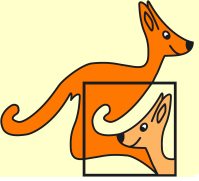
Solution. For each $b \in B$ let $N(b) = \{n \in \mathbb{N} : \ g(n) = b\}$. Observe, that the sets $N(b)$, $b \in B$, cover all the set $\mathbb{N}$. In each $N(b)$ find the smallest element n_b. Define function $f : B \to \mathbb{N}$ by $f(b) = n_b$. Clearly f is one-to-one. By Problem 3, B is countable. □

Remark. To show that a set is countable, it is enough to arrange all the elements of this set into a sequence. Such a sequence will be a function from $\mathbb{N}$ onto the given set.



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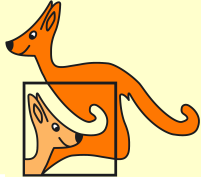

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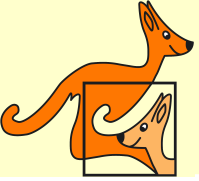

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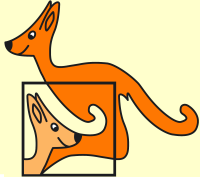

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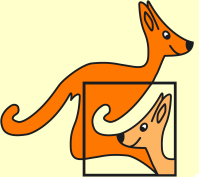

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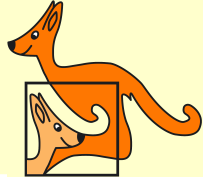

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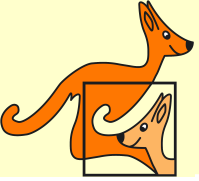

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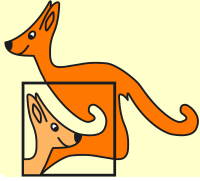

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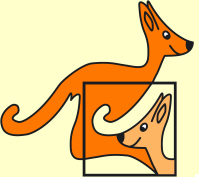
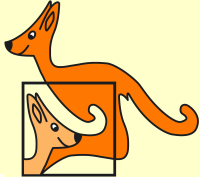


Subsets of a given set



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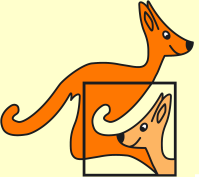

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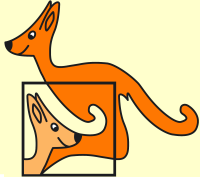

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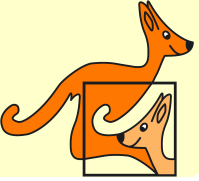

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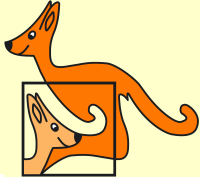

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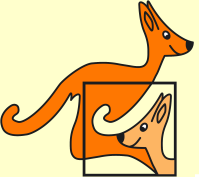

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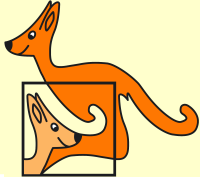

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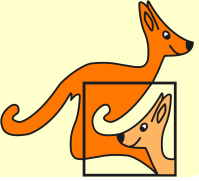

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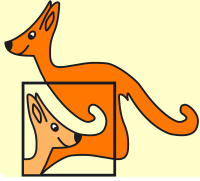

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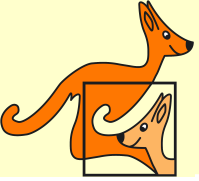

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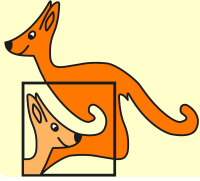

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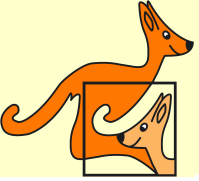

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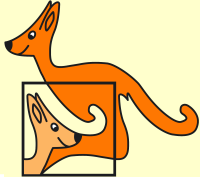

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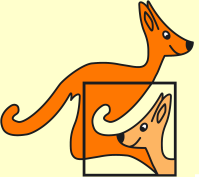

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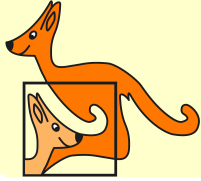

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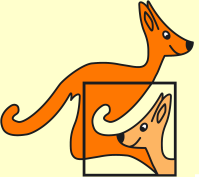

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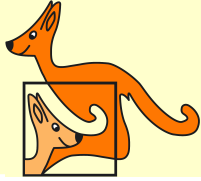

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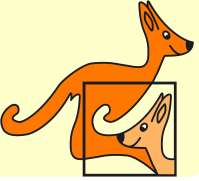

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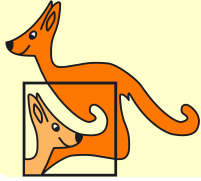

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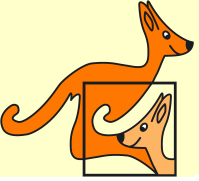

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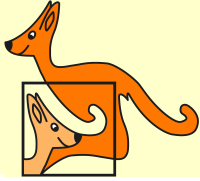

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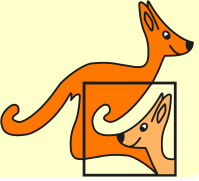

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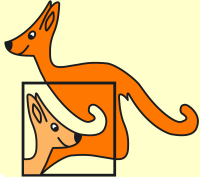

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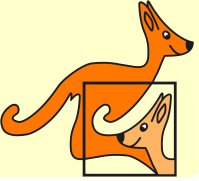

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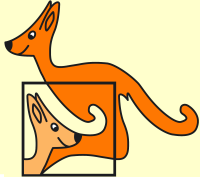

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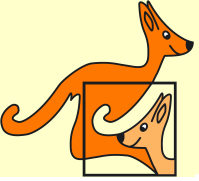

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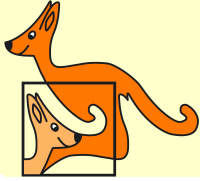

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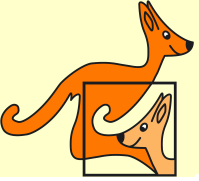

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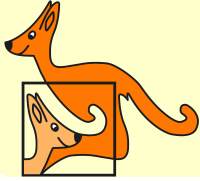

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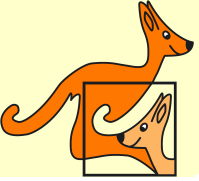
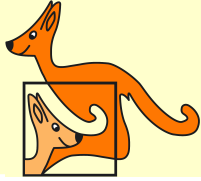


An interesting theorem



Theorem. *Let X be a set. Then:*

(*i*) *There is a one-to-one function $I : X \to 2^X$.*

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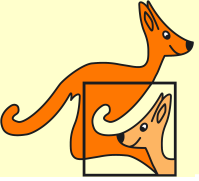

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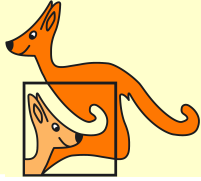

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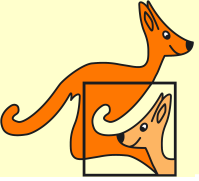
Proof. Ad (i) Define the function $I : X \to 2^X$ by $I(x) = \{x\}$. Clearly I is one-to-one (not onto).



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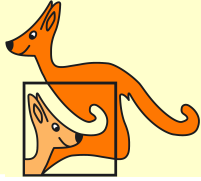

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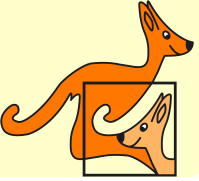

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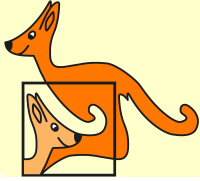

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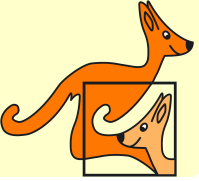

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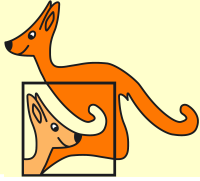

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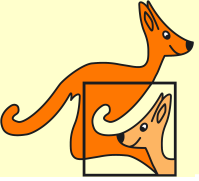

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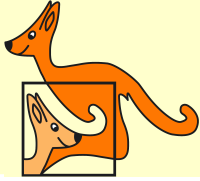

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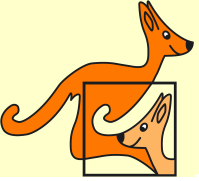

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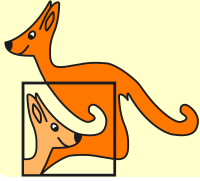

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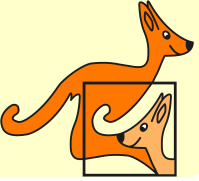

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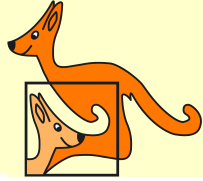

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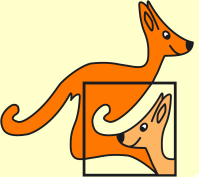
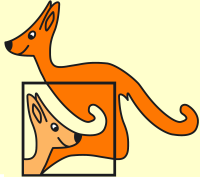
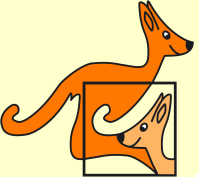
In each case we obtained a contradiction, which means that the assumption about the existence of bijection was false. This completes the proof of the theorem. □



Are all infinite sets countable?





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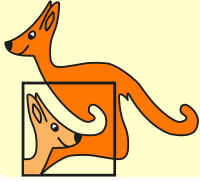

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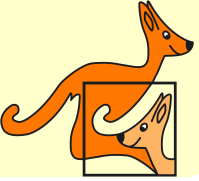

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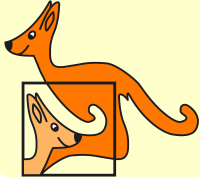

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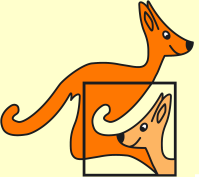

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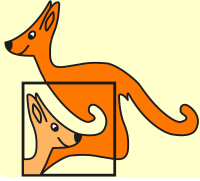

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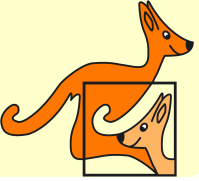
Are there other uncountable sets? Yes, we will soon meet some of them.



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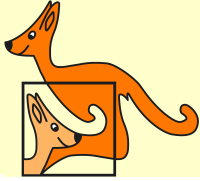

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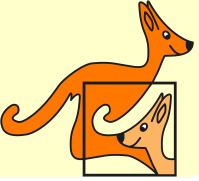

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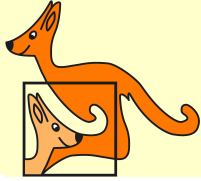

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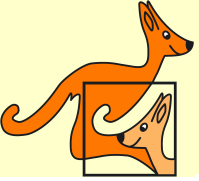

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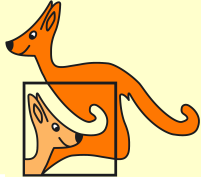

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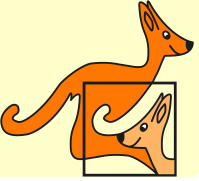

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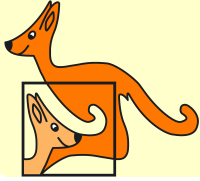

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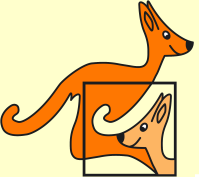

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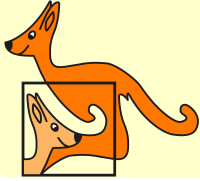

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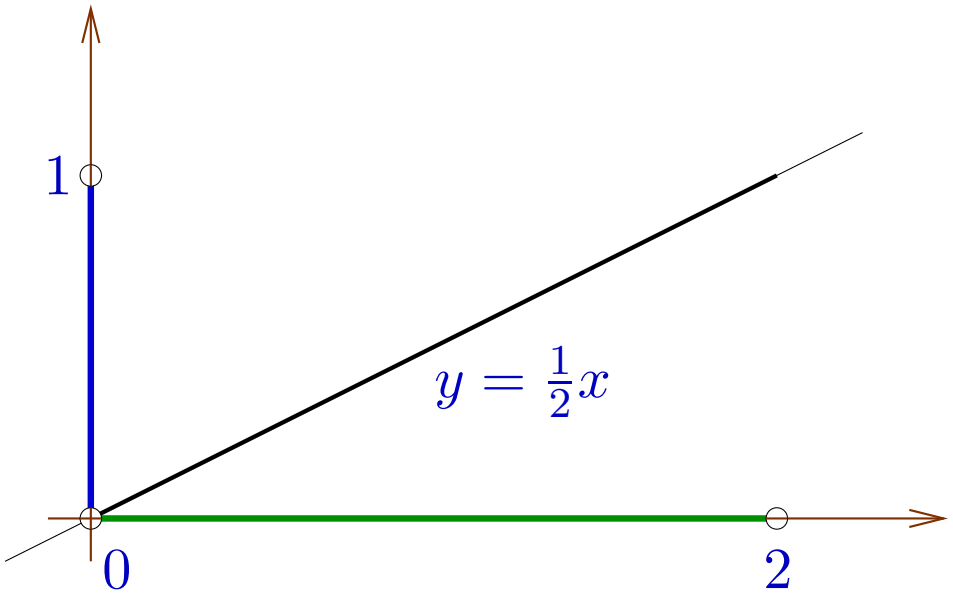

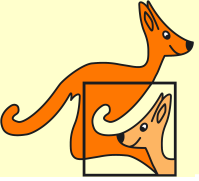

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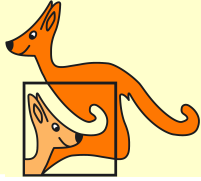

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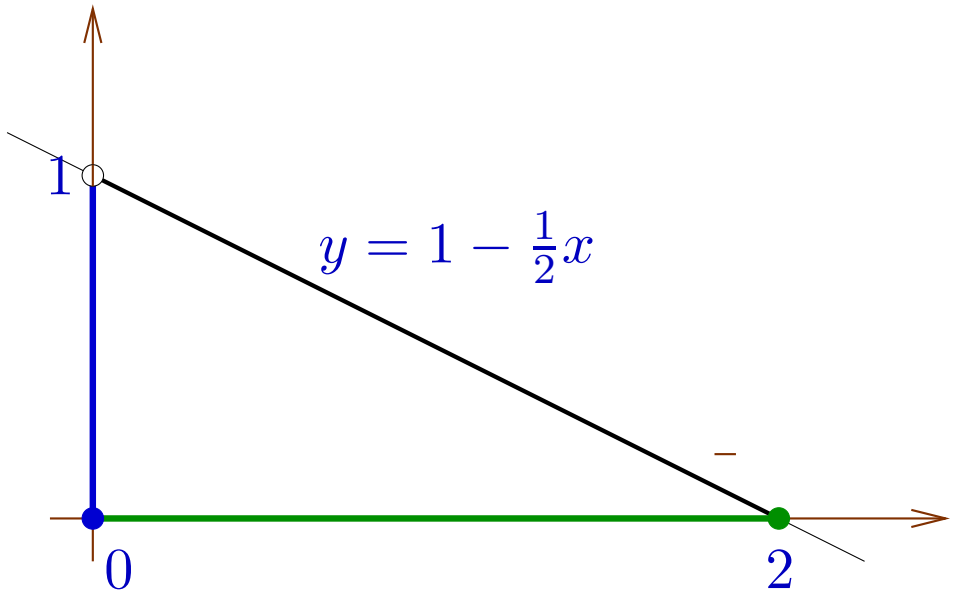

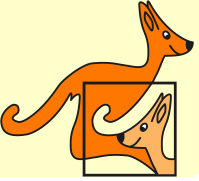

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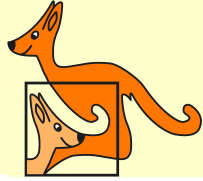

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Corollary. *All open line segments on* $\mathbb{R}$ *are equinumerous.*

All half-open line segments on $\mathbb{R}$ *are equinumerous.*

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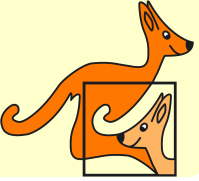

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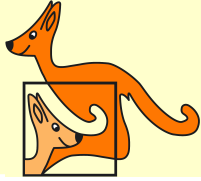

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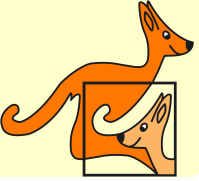

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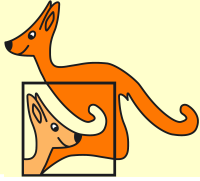

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Why f is one-to-one?

Suppose $f(x_1) = f(x_2)$, i.e. $\frac{x_1}{1-|x_1|} = \frac{x_2}{1-|x_2|}$. Observe that $x_1 \geqslant 0$ if and only if (iff) $x_2 \geqslant 0$. Thus for $x_1, x_2 \geqslant 0$ we have $\frac{x_1}{1-x_1} = \frac{x_2}{1-x_2}$, while for $x_1, x_2 < 0$ we have $\frac{x_1}{1+x_1} = \frac{x_2}{1+x_2}$.

Easy calculations show that in both cases $x_1 = x_2$.

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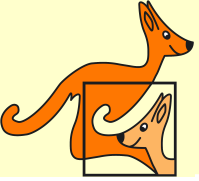
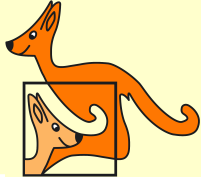
Why f is onto?

First observe that for $x \in (-1, 1)$, $x \geqslant 0$ iff $f(x) \geqslant 0$.

Now take any $y \in \mathbb{R}$. We seek such an x, that $f(x) = y$, i.e. $\frac{x}{1-x} = y$ for $y \geqslant 0$ and $\frac{x}{1+x} = y$ for $y < 0$.

Easy calculations show that $x = \frac{y}{1+y}$ for $y \geqslant 0$ and $x = \frac{y}{1-y}$ for $y < 0$.





Problem 6. Show, that line segment $(-1, 1)$ and $\mathbb{R}$ are equinumerous.

Solution. Consider function $f : (-1, 1) \to \mathbb{R}$ given by

$$f(x) = \frac{x}{1 - |x|}.$$

We will show that f is a bijection.

Thus f is a bijection.

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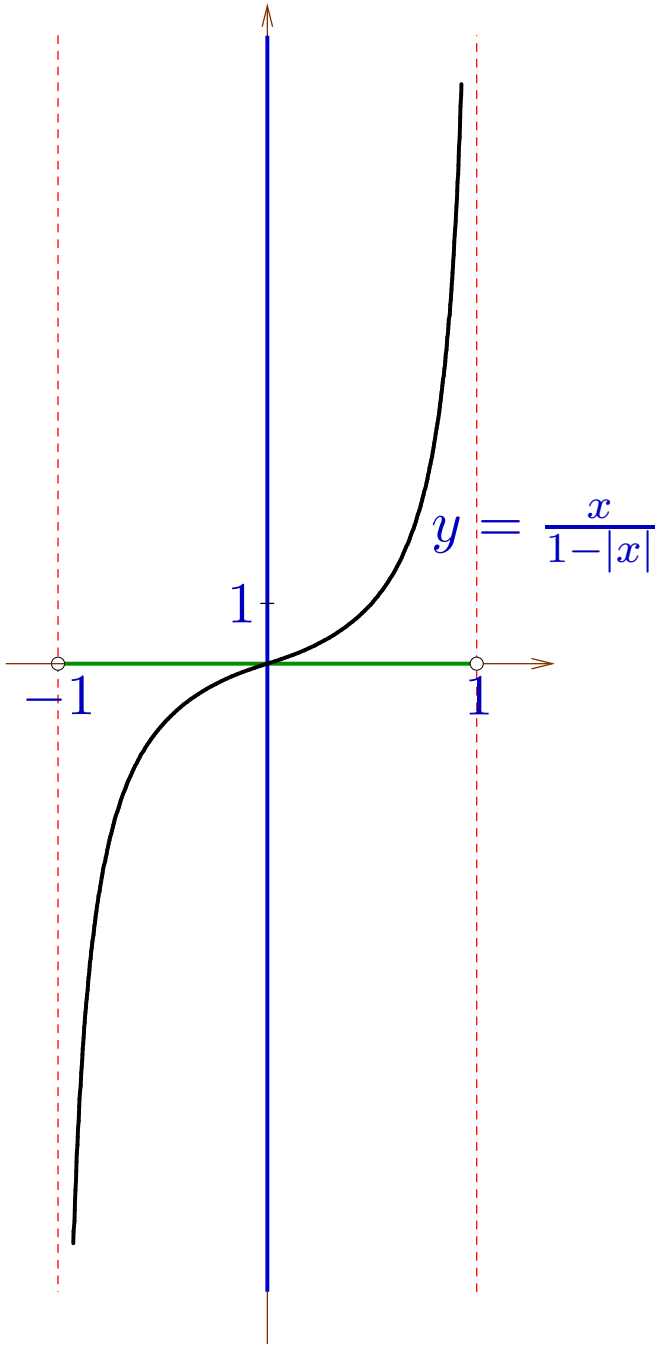
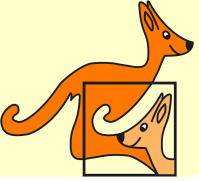
We have proved that the line segment $(-1, 1)$ and $\mathbb{R}$ are equinumerous. $\square$

Intervals





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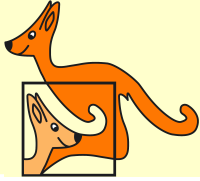
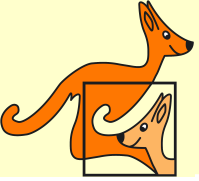


Problem 7. Extract x from the formula $f(x) = \frac{x}{1-|x|}$, to get the inverse function $g : \mathbb{R} \to (-1, 1)$.



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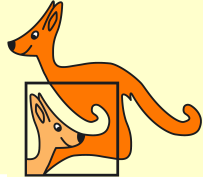

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$$f(x) = \begin{cases} \frac{x}{1-x} & \text{for } x \geqslant 0, \\ \frac{x}{1+x} & \text{for } x < 0. \end{cases}$$

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Finally get $x = g(y) = \frac{y}{1+|y|}$ for $y \in \mathbb{R}$.

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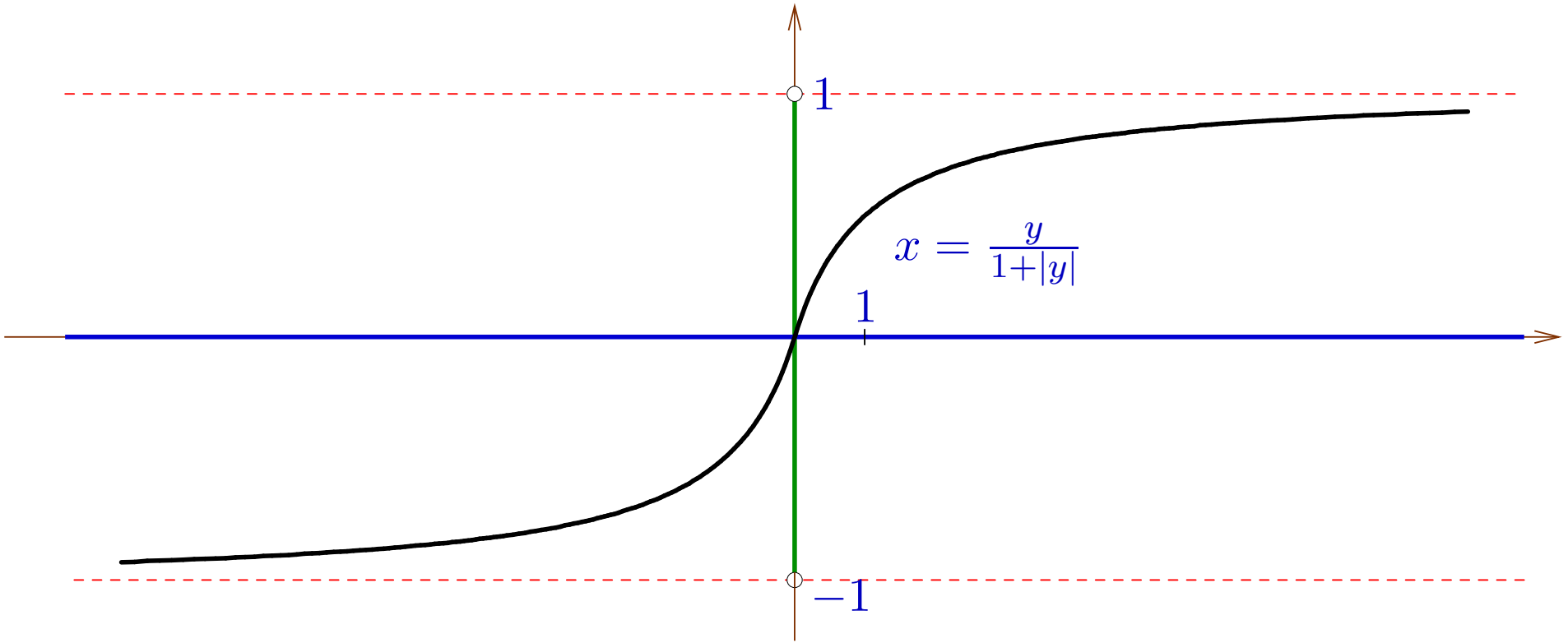
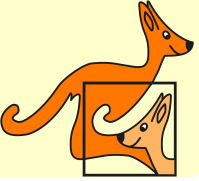
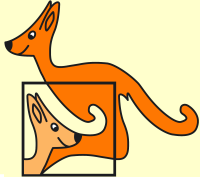
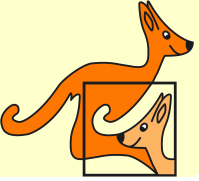




Cardinal numbers



We assign to each set A an object called a *cardinal number*, denoted by $\mathrm{card}(A)$, and we put some conditions, as follows.



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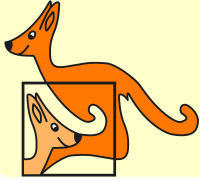

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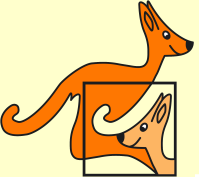

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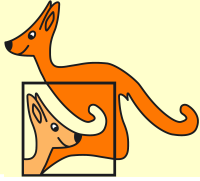

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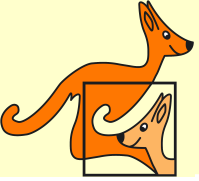

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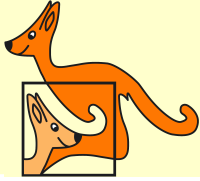

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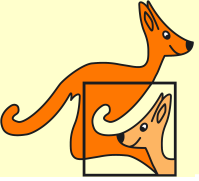

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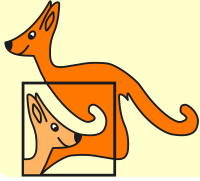

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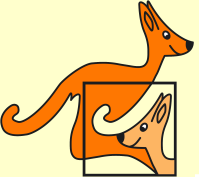

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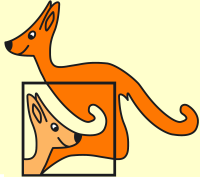

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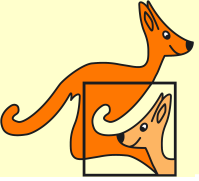
C5. For any collection of sets the corresponding collection of their cardinal numbers contains the smallest element with respect to the relation $\leqslant$.



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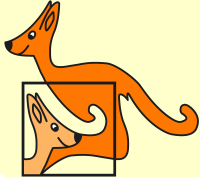

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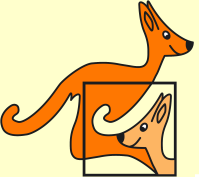
If $\mathrm{card}(A) \leqslant \mathrm{card}(B)$ and $\mathrm{card}(A) \neq \mathrm{card}(B)$, then we write $\mathrm{card}(A) < \mathrm{card}(B)$ and say that *card*(A) is *smaller* then $\mathrm{card}(B)$, or that $\mathrm{card}(B)$ is *greater* than $\mathrm{card}(A)$.

For example, $\mathrm{card}(\{0,1,2,3\})$ is smaller than $\mathrm{card}(\mathbb{N})$, $\mathrm{card}(\{0,1,2,3\}) < \mathrm{card}(\mathbb{N})$.



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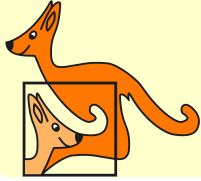

Theorem. *Let A and B be sets. The following conditions are equivalent.*

(i) $\mathrm{card}(A) \leqslant \mathrm{card}(B)$.

(ii) *There is a one-to-one function* $f : A \to B$.

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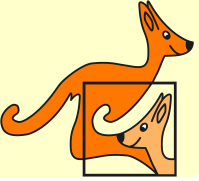

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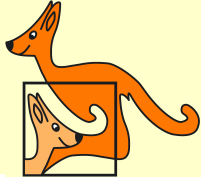

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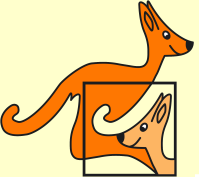

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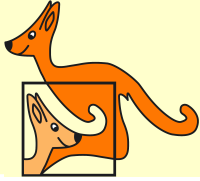

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Suppose that (ii) is true and let $f : A \to B$ be one-to-one function. Then $f(A) \subset B$ and there is one-to-one and onto function $f^{-1} : f(A) \to A$. Fix any $a \in A$. Define function $g : B \to A$ by

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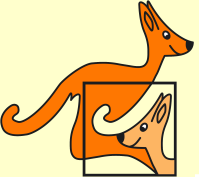
Clearly g is onto.



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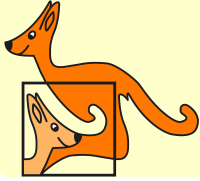

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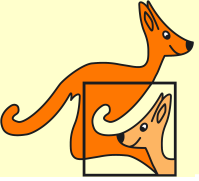

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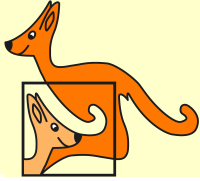

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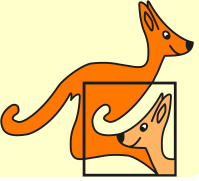
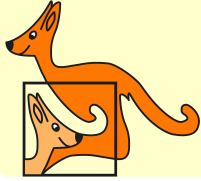
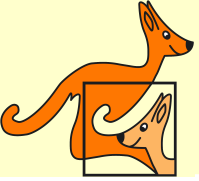
By C4, $\mathrm{card}((-1,1)) = \mathrm{card}(\langle 0,1)) = \mathrm{card}(\langle 0,1\rangle)$. □



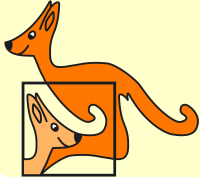
An important corollary



Corollary. *The set* $\mathbb{R}$ *and all line segments are equinumerous.*

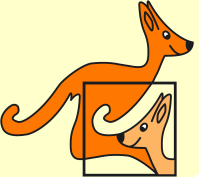


Periodic fractions



Let us recall the concept of a periodic fraction. When dividing natural numbers by natural numbers, it sometimes happens that infinitely non-zero digits appear after the decimal point, e.g. $1:3=0.33333\ldots$, $1:11=0.09090909\ldots$, $5:12=0.41666666\ldots$. In the given examples, digits or groups of digits repeat from a certain point: in the quotient $1:3$ we see only 3, in the quotient $1:11$ the group of digits 09 repeats, and in the quotient $5:12$ the digit 6 is repeated. We call such repeating groups of digits a *period*. In writing, we put the period in parenthesis:

$$1:3=0.3333333\ldots=0.(3),$$
$$1:11=0.09090909\ldots=0.(09),$$
$$5:12=0.41666666\ldots=0.41(6).$$



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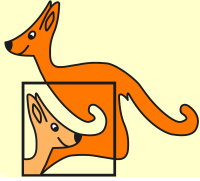

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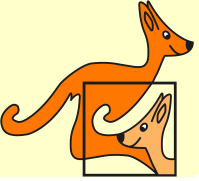
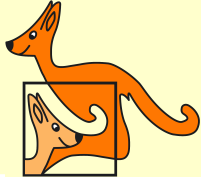
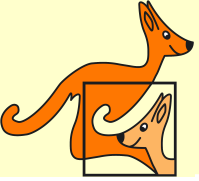
If a decimal fraction expresses the quotient of natural numbers (i.e. a rational number), it is either finite (from a certain point it contains only zeros) or periodic. Representation a rational number as a decimal is not always unique. For instance $0.(9)=1.0$, $0.2(9)=0.3$, $0.5=0.4(9)$, $0.01=0.00(9)$.



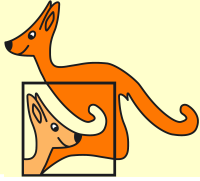
Decimal fractions



Theorem. *Every non-zero real number has a unique decimal representation with the property that it contains an infinite many of non-zero digits.*



Are $\mathbb{N}$ and $\mathbb{R}$ equinumerous?



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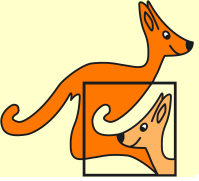

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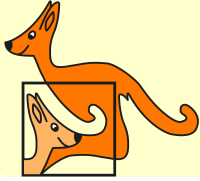

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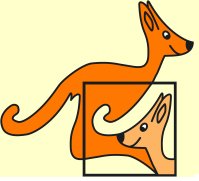

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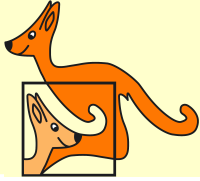

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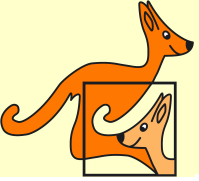
Write each number from $\langle 0, 1\rangle$ in decimal representation with infinitely many non-zero digits (and $0 = 0.(0)$). Denote

$$x_n = 0\,.\,x_{n,1}\,x_{n,2}\,x_{n,3}\,x_{n,4}\,x_{n,5}\,\ldots, \quad n \geqslant 0,$$

where $x_{n,k} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $n \geqslant 0$, $k \geqslant 1$.



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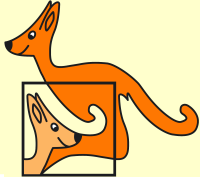

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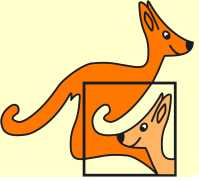

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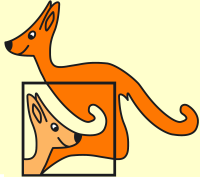

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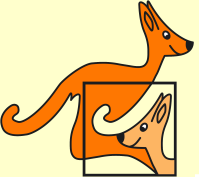
We will show, that such a table can not contain all elements of $\langle 0, 1\rangle$. To do this we will construct a sequence of digits that is an element of $\langle 0, 1\rangle$ which is not in the above table.



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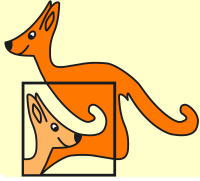

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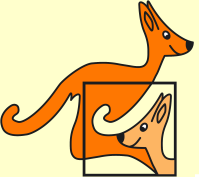
Let's look at the diagonal part of the above table to the right of the decimal point.



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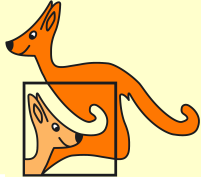

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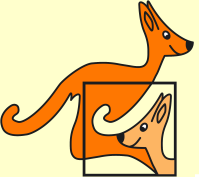

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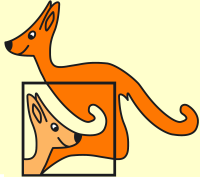

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\end{aligned}
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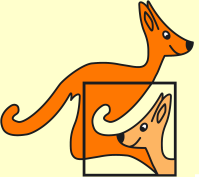
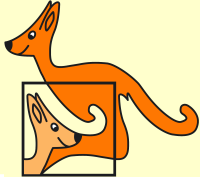
Define the sequence $\overline{y} \in \langle 0, 1\rangle$ by creating it from the consecutive digits marked in red: $\overline{y} = 0\,.\,x_{0,1}\,x_{1,2}\,x_{2,3}\,x_{3,4}\,x_{4,5}\,x_{5,6}\,x_{6,7}\ldots$



Are $\mathbb{N}$ and $\mathbb{R}$ equinumerous?



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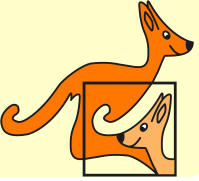

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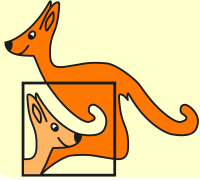

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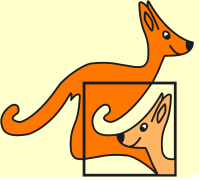
Observe, that y is not in the table above, since $y_n \neq x_{n,n+1}$, $n \geqslant 0$. We got a contradiction, so $\mathbb{N}$ and $\langle 0,1\rangle$ are not equinumerous. In particular, $\mathbb{R}$ is uncountable. In other words, $\mathrm{card}(\mathbb{N}) < \mathrm{card}(\mathbb{R})$.



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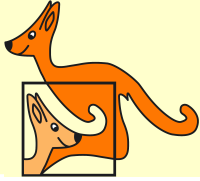

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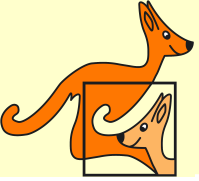

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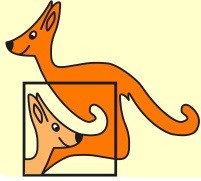

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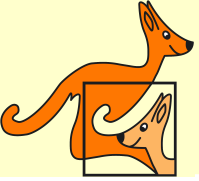

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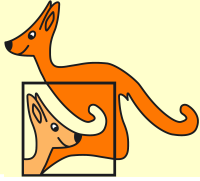

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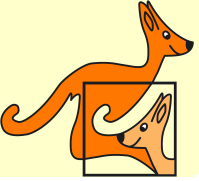
Thus $\mathrm{card}(\{0,1\}^{\mathbb{N}}) = \mathrm{card}(2^{\mathbb{N}})$.



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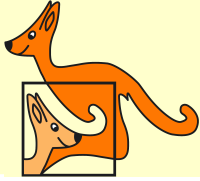

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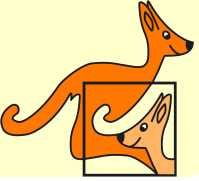
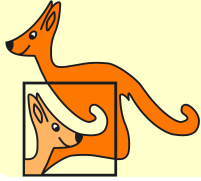
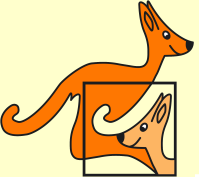
We will show that $\mathrm{card}(\{0,1\}^{\mathbb{N}}) = \mathrm{card}(\langle 0,1\rangle) = \mathrm{card}(\mathbb{R})$.



Uncountable sets



Problem 9. Show that $\mathrm{card}(\{0,1\}^{\mathbb{N}}) = \mathrm{card}(\langle 0,1\rangle)$.



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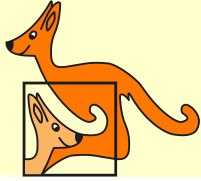

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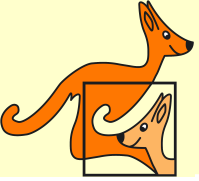

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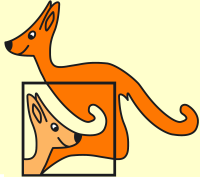

Problem 9. Show that $\mathrm{card}(\{0,1\}^{\mathbb{N}}) = \mathrm{card}(\langle 0,1\rangle)$.

Solution.

We will define one-to-one function $w : \{0,1\}^{\mathbb{N}} \to \langle 0,1\rangle$ (which will imply that $\mathrm{card}(\{0,1\}^{\mathbb{N}}) \leqslant \mathrm{card}(\langle 0,1\rangle)$), and onto function $p : \{0,1\}^{\mathbb{N}} \to \langle 0,1\rangle$ (which will imply that $\mathrm{card}(\langle 0,1\rangle) \leqslant \mathrm{card}(\{0,1\}^{\mathbb{N}})$). By C4 we obtain equality.

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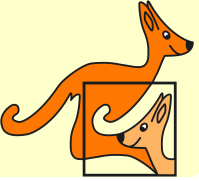
Solution.

Let $w : \{0,1\}^{\mathbb{N}} \to \langle 0,1\rangle$,

$$w((a_n)_{n\geqslant 0}) = \frac{a_0}{3} + \frac{a_1}{9} + \frac{a_2}{27} + \cdots + \frac{a_n}{3^{n+1}} + \cdots .$$

It is clear that w is one-to-one (why it is clear?).

Therefore $\mathrm{card}(\{0,1\}^{\mathbb{N}}) \leqslant \mathrm{card}(\langle 0,1\rangle)$.



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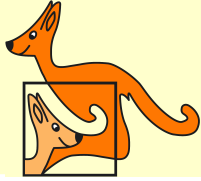

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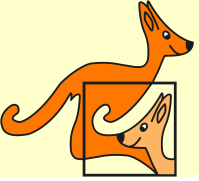
Solution.

Let $p : \{0,1\}^{\mathbb{N}} \to \langle 0,1\rangle$,

$$p((a_n)_{n\geqslant 0}) = \frac{a_0}{2} + \frac{a_1}{4} + \frac{a_2}{8} + \cdots + \frac{a_n}{2^{n+1}} + \cdots .$$

The function p is onto because every number from the interval $\langle 0,1\rangle$ has a binary expansion starting from zero point.

This implies that $\mathrm{card}(\langle 0,1\rangle) \leqslant \mathrm{card}(\{0,1\}^{\mathbb{N}})$



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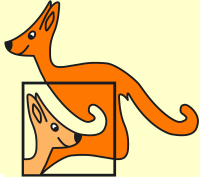

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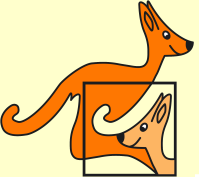
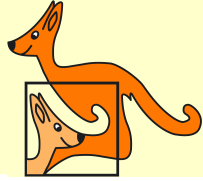
Solution.

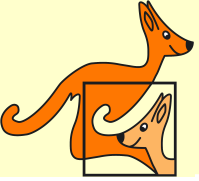
We got $\mathrm{card}(\{0,1\}^{\mathbb{N}}) \leqslant \mathrm{card}(\langle 0,1\rangle)$ and $\mathrm{card}(\langle 0,1\rangle) \leqslant \mathrm{card}(\{0,1\}^{\mathbb{N}})$, so $\mathrm{card}(\{0,1\}^{\mathbb{N}}) = \mathrm{card}(\langle 0,1\rangle) = \mathrm{card}(\mathbb{R})$.



Continuum



The cardinal number $\mathrm{card}(\mathbb{R})$ is called *continuum* and is denoted by $\mathfrak{c}$, i.e. $\mathrm{card}(\mathbb{R}) = \mathfrak{c}$.



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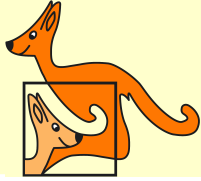

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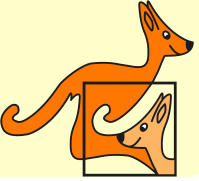

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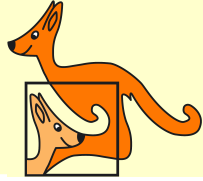

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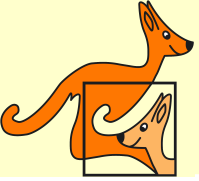

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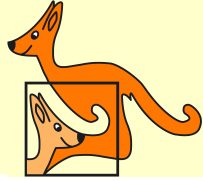

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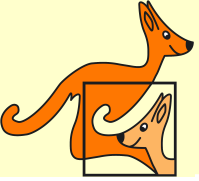
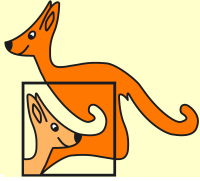
In mathematics by continuum we understand also a compact connected set. A typical example of such a set is any closed line segment $\langle a, b\rangle$, $a < b$. Clearly $\mathrm{card}(\langle a, b\rangle) = \mathfrak{c}$.



Continuum



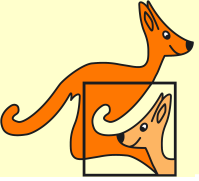
The cardinal number $\mathrm{card}(\mathbb{R})$ is called *continuum* and is denoted by $\mathfrak{c}$, i.e. $\mathrm{card}(\mathbb{R}) = \mathfrak{c}$.

The word "continuum" comes from the Latin *continuus* which means continuous, uninterrupted.

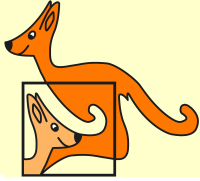
The word in general means a continuous, ordered set of an infinite number of elements flowing into one another.

In mathematics by continuum we understand also a compact connected set. A typical example of such a set is any closed line segment $\langle a, b\rangle$, $a < b$. Clearly $\mathrm{card}(\langle a, b\rangle) = \mathfrak{c}$.

Bolzano, already known to us, understood the continuum in a way that can now be formulated as follows. The continuum is a whole composed of (spatial, temporal or material) points at which for each point there is a specific great distance, so that in this and every one shorter distance there is a point belonging to this whole.



Other cardinal numbers



The cardinal number $\mathrm{card}(\mathbb{N})$ is denoted by $\aleph_0$ — we pronounce it as *aleph zero*. Aleph $\aleph$ is the first letter of the Hebrew alphabet. This notation was introduced into mathematics by Georg Cantor[3].

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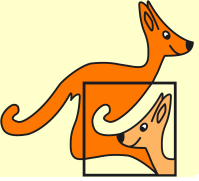

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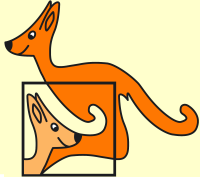

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Cardinal numbers have the following property (it results from C5).

C6. Every cardinal number has a successor, i.e. if $\mathfrak{m}$ is a cardinal number, then there is a cardinal number $\mathfrak{n}$ such that $\mathfrak{m} < \mathfrak{n}$ and every other cardinal number $\mathfrak{p} > \mathfrak{m}$ satisfies $\mathfrak{n} \leqslant \mathfrak{p}$.

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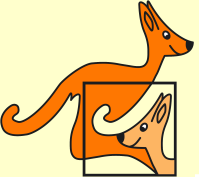

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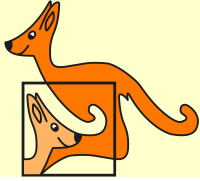

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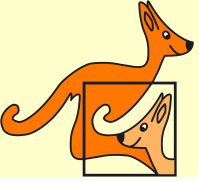

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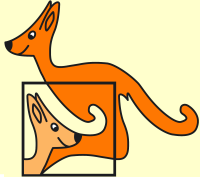

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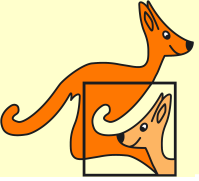

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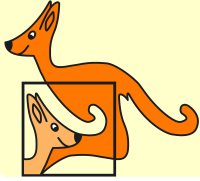

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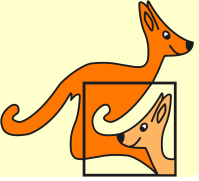

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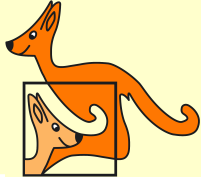

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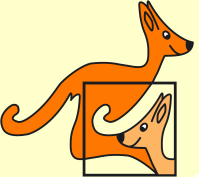

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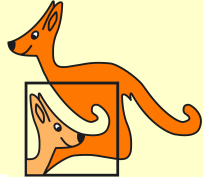

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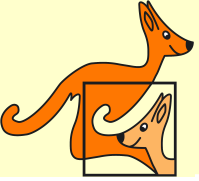

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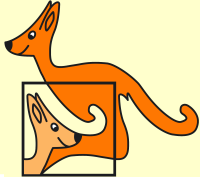

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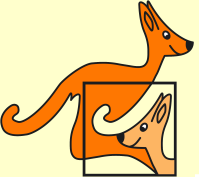

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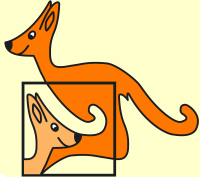

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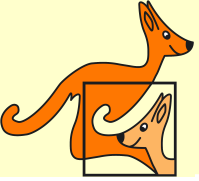

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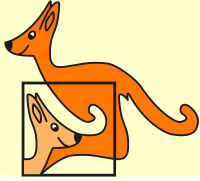

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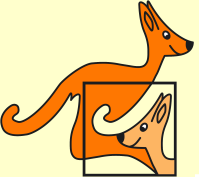
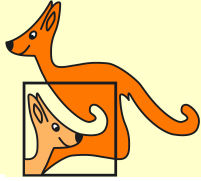
Suggestion. The concept of an object containing all the cardinal numbers (whatever those may be) can be considered an example of potential infinity.



Is $\mathfrak{c}$ equal to $\aleph_1$?



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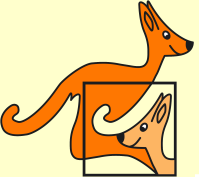

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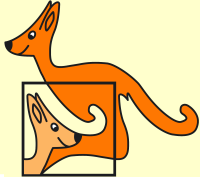

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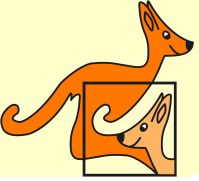

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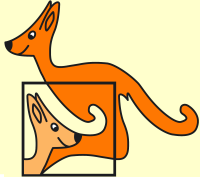

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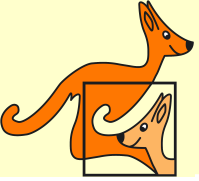

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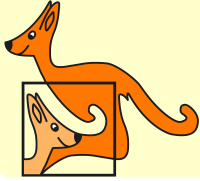

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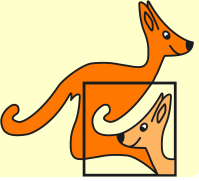
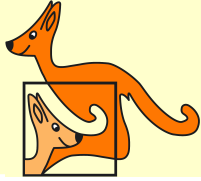
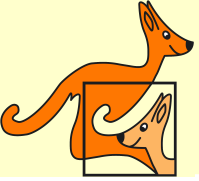
Note, that *consistent theory* is a theory without contradictions.



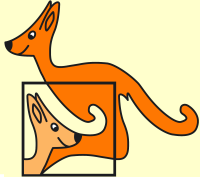
And what is ∞?



The infinity symbol ∞ was proposed by John Wallis (23.11.1616 – 28.10.1703, English mathematician), but in the sense of potential infinity, and the symbol $\frac{1}{\infty}$ as an infinitesimal quantity.



And what is ∞?



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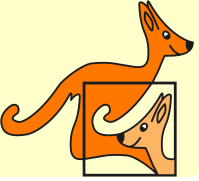

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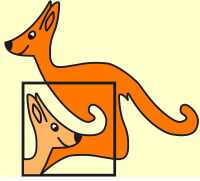

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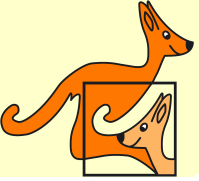
Let us consider some example.



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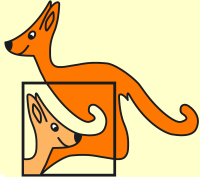
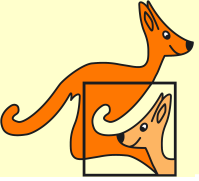


Consider line segment $(-1, 1)$ and the real line:

Both lines have no ends.



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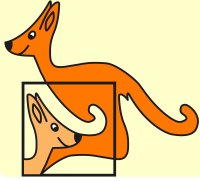
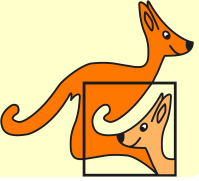


Let's add its ends to the segment — we will get a closed segment.



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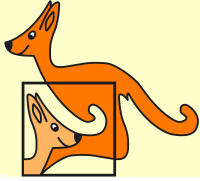
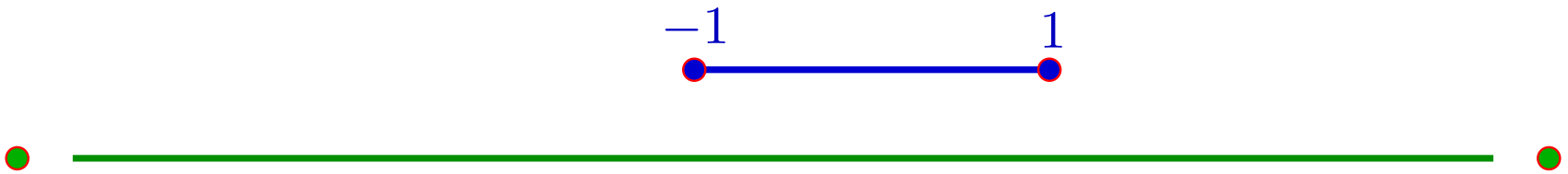
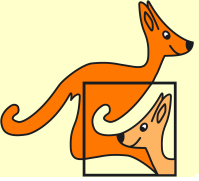


And why can't we add "ends" to the real line? Let's do it!





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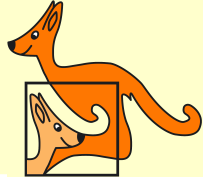
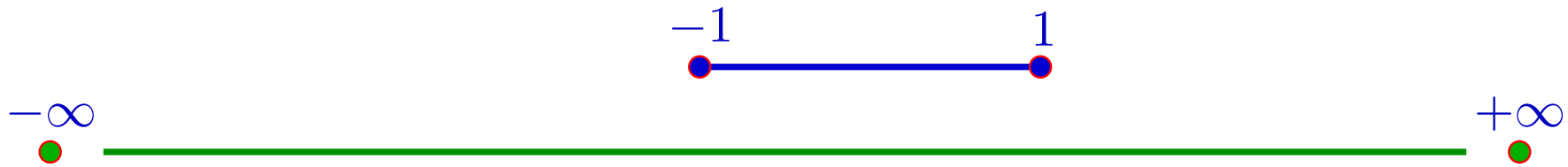
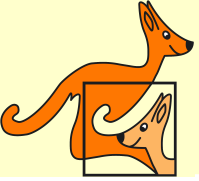


Let's give these "ends" names:





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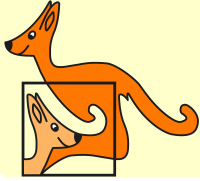

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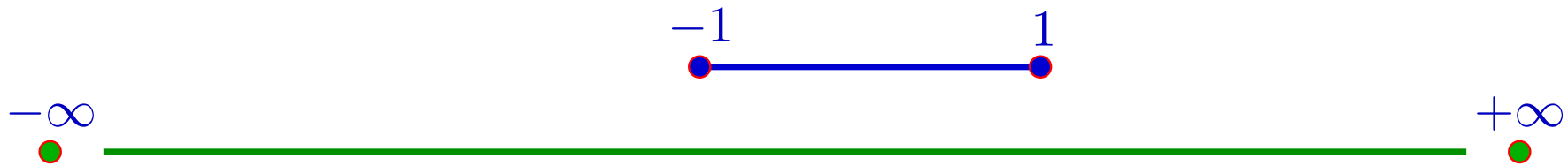
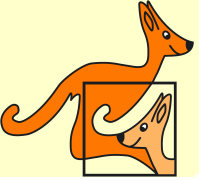
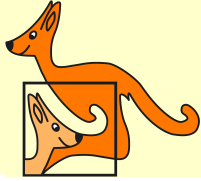


Thus we have ∞ ;)

The (happy) end